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A METHOD OF VERIFIED COMPUTATIONS FOR SOLUTIONS TO SEMILINEAR PARABOLIC EQUATIONS USING SEMIGROUP THEORY*

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Abstract. This paper presents a numerical method for verifying the existence and local uniqueness of a solution for an initial-boundary value problem of semilinear parabolic equations. The main theorem of this paper provides a sufficient condition for a unique solution to be enclosed within a neighborhood of a numerical solution. In the formulation used in this paper, the initial-boundary value problem is transformed into a fixed-point form using an analytic semigroup. The sufficient condition is derived from Banach's fixed-point theorem. This paper also introduces a recursive scheme to extend a time interval in which the validity of the solution can be verified. As an application of this method, the existence of a global-in-time solution is demonstrated for a certain semilinear parabolic equation.

Key words. semilinear parabolic initial-boundary value problems, verified numerical computations, existence and local uniqueness

AMS subject classifications. 65M60, 65M15, 35K20

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1. Introduction.

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. We consider the following initial-boundary value problem of semilinear parabolic equations:

$$(1)\qquad \begin{cases} \partial_t u + Au = f(u) & \text{in } J\times\Omega,\\ u(t,x)=0 & \text{on } J\times\partial\Omega,\\ u(t_0,x)=u_0(x) & \text{in } \Omega. \end{cases}$$

Here, f is a mapping from $\mathbb{R}$ to $\mathbb{R}$ such that $f\circ\phi\in L^2(\Omega)$ for each $\phi\in H_0^1(\Omega)$, so we can also consider f as a map from $H_0^1(\Omega)$ to $L^2(\Omega)$ given by $\phi\mapsto f\circ\phi$, and we impose that this be twice Fréchet differentiable. Furthermore, $J:=(t_0,t_1]$ with $0\le t_0<t_1<\infty$ or $J:=(0,\infty)$, $\partial_t=\frac{\partial}{\partial t}$, $u_0\in H_0^1(\Omega)$ is a given initial function, and $A: D(A)\subset H^\alpha(\Omega)\cap H_0^1(\Omega)\to L^2(\Omega)$ is defined by

$$A := -\sum_{1\le i,j\le 2}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right),$$

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where $a_{ij}(x) \in W^{1,\infty}(\Omega)$, $a_{ij}(x) = a_{ji}(x)$, and α depends on the shape of the domain.[1] The operator A is also assumed to satisfy the following ellipticity condition:

$$\sum_{1\leq i,j\leq 2} a_{ij}(x)\xi_i\xi_j \geq \mu\|\xi\|^2 \quad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^2 \text{ with } \mu > 0,$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector. It is known [23, 29] that the operator "$-A$" generates an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ over $L^2(\Omega)$. This initial-boundary value problem includes certain reaction-diffusion equations and nonlinear heat equations.

The main aim of this paper is to present Theorem 2.7, which provides a sufficient condition for guaranteeing the existence and local uniqueness of a mild solution for (1) when $J = (t_0, t_1]$ with $0 \leq t_0 < t_1 < \infty$. The definition of this mild solution is given in section 2.1. Such a sufficient condition can be checked using verified numerical computations, which derive mathematically rigorous conclusions using numerical computations.

To extend the time interval in which the existence of the mild solution is guaranteed, we consider the initial-boundary value problem (1) with $J = (t_1, t_2]$ ($t_1 < t_2 < \infty$). For this purpose the initial function should be replaced by a certain ball in $H_0^1(\Omega)$ that is an inclusion of $u(t_1, \cdot)$. Then, we can apply Theorem 2.7 for this initial-boundary value problem on $(t_1, t_2]$ and recursively repeat this process. If the sufficient condition of Theorem 2.7 does not hold, then the process is stopped.

Considering semilinear parabolic equations of the form (1), many analytic studies have been performed using the semigroup theory,[2] which originated from pioneering studies by Hille [12] and Yosida [31]. For example, the following results concerning nonnegative solutions are well known:

- In the case that $f(u) = u^p$ ($p > 1$) and $\Omega = \mathbb{R}^m$ ($m \in \mathbb{N}$) in (1), Fujita determined an exponent concerning the existence of a global-in-time solution[3] in [7]. This is the so-called Fujita exponent. It has also been shown in [16] that the Fujita exponent of (1) is determined by the minimal eigenvalue of A, which depends on the domain Ω.
- Global existence in the case of small initial data has also been shown in, e.g., [13] for (1), where $f(0) = 0$. In such a problem, the solution exponentially converges to the zero function as $t \to \infty$.
- When $f(u) = u^p$ ($q \geq p > 1$) in (1), the existence of a mild L^q-solution has been studied in [1]. Here a mild L^q-solution of (1) is defined by a function $u \in C(J; L^q(\Omega))$ that is given by

$$u(t) = e^{-(t-t_0)A}u_0 + \int_{t_0}^{t} e^{-(t-s)A} f(u(s))ds.$$

  Let $q_c = p - 1$. If $q \geq q_c$ (and $q > p$ if $q_c = q$), then there exists $T > t_0$ such that this problem admits a unique mild L^q-solution for $J = [t_0, T)$.

For numerous other results, see, e.g., [2, 25].

[1] The expression $A\phi$ for $\phi \in H_0^1(\Omega)$ is defined in the *distributional* sense, i.e., $A\phi \in H^{-1}(\Omega)$ for $\phi \in H_0^1(\Omega)$, and one defines $D(A) := \{\phi \in H_0^1(\Omega) : A\phi \in L^2(\Omega)\}$. Furthermore, it can be proved (see, e.g., [11]) that $D(A) \subset H^{\alpha}(\Omega)$ for some α depending on the domain, e.g., $\alpha \in (\frac{3}{2}, 2]$ when Ω is polygonal.

[2] For the development of the semigroup theory, see [23].

[3] This refers to solutions of (1) whose existence is proved for $t \in (0, \infty)$.

The results obtained in the analytical studies above give qualitative properties of solutions to (1). On the other hand, in order to observe the behavior of solutions, various numerical schemes have been developed for this type of parabolic equations (cf. [28]). Furthermore, convergence analyses have been performed for such numerical schemes. For example, Fujita and his collaborators [8, 10, 9] have demonstrated the convergence of a full discretization scheme using semigroup theory. They have also presented a priori error estimates of the scheme. Subsequently, some a posteriori error estimates based on Green's functions and elliptic reconstructions have been introduced in [5, 15].

A number of computer-assisted methods based on verified numerical computations for proving the existence and local uniqueness of solutions to various elliptic equations have been developed over the last two decades by Nakao, Plum, and their collaborators (see [19, 22, 24, 27, 30] and references therein). Recently, such computer-assisted methods have been extended to parabolic equations in [14, 20, 21]. In [21], Nakao, Kinoshita, and Kimura proposed a method for estimating the norms of inverse operators for linear parabolic differential equations. Next, in [20], the same authors proposed the *"time interpolation scheme"* for a class of heat equations and provided a constructive error estimate for this scheme. Subsequently, in [14] the norm estimate proposed in [21] was improved using the error estimate proposed in [20].

Another recent approach to verified numerical computations for partial differential equations is based on the Conley index and the verification of corresponding topological conditions. See, e.g., [33, 32]. These studies use an astute application of the characteristics of the spectrum method. In addition, in [4] Day, Lessard, and Mischaikow made use of the spectrum method to propose a method for finding a continuous parameterized family of stationary solutions for certain evolution equations. For the evolution equation itself, Zgliczyński proved the existence of periodic solutions to the Kuramoto–Sivashinsky equation defined on a one-dimensional space in [32].

The main contribution of this paper is to combine a "classical analysis" with "computer-assisted methods" to provide a new numerical method of bounding a solution for semilinear parabolic equations using classical semigroup theory. Computer-assisted methods that employ analytic semigroups have not been applied in any previous studies. The combination of quantitative estimates arising from verified numerical computations and qualitative results obtained by analytical studies can be expected to provide a good approach to many unsolved problems. Moreover, the existing studies [14, 20, 21] require a computable a priori error estimate (called a constructive error estimate), which is related to an orthogonal projection. This error estimate restricts the function space of approximate solutions and numerical schemes. On the other hand, in this paper, the projection error estimate is required only for obtaining the minimal eigenvalue of A. This indicates that our method can employ more accurate approximate solutions using some numerical schemes.

The remainder of this paper is organized as follows. In section 2, we introduce our notation and present the definition of a mild solution for (1). Furthermore, we transform the initial-boundary value problem (1) into a fixed-point form using an analytic semigroup formulation. Then, on the basis of Banach's fixed-point theorem, we derive a sufficient condition for the existence and local uniqueness of a mild solution. If this condition holds, then the mild solution is enclosed in a ball centered at a numerical solution ω with a radius $\rho > 0$:

$$B_J(\omega, \rho) := \left\{ y \in L^\infty\left(J; H_0^1(\Omega)\right) : \|y - \omega\|_{L^\infty\left(J; H_0^1(\Omega)\right)} \le \rho \right\},$$

where

$$L^\infty\left(J; H_0^1(\Omega)\right) := \left\{u : J \times \Omega \to \mathbb{R}, u(t,\cdot) \in H_0^1(\Omega) : \operatorname*{ess\,sup}_{t\in J} \|u(t,\cdot)\|_{H_0^1} < \infty\right\}$$

is a Banach space with the norm $\|u\|_{L^\infty(J;H_0^1(\Omega))} := \operatorname{ess\,sup}_{t\in J} \|u(t,\cdot)\|_{H_0^1}$. This sufficient condition is presented in Theorem 2.7. The proof of Theorem 2.7 and some related estimates are also provided. Afterwards, we set $0 \le t_0 < t_1 < \cdots < t_n < \infty$ for a fixed natural number n, and we define $J_k := (t_{k-1}, t_k]$ $(k = 1, 2, \ldots, n)$ and $J := \cup J_k$. In section 3, we consider the initial-boundary value problem (1). A recursive scheme for enclosing the mild solution for $t \in J$ is introduced in section 3.1. For $k = 1, 2, \ldots, n$, $B_{J_k}(\omega|_{t\in J_k}, \rho_k)$ becomes an enclosure of the mild solution $u(t,\cdot)$, $t \in J_k$. We construct each enclosure using an iterative numerical verification scheme based on Theorem 2.7. Then, we demonstrate numerically that the mild solution for $t \in J$ uniquely exists in

$$B(\omega) := \left\{y \in L^\infty\left(J; H_0^1(\Omega)\right) :\ y|_{t\in J_k} \in B_{J_k}(\omega|_{t\in J_k}, \rho_k) \text{ for } k = 1, 2, \ldots, n\right\}. \tag{2}$$

In section 3.2, we present some computational examples to illustrate the features of the numerical verification method. As an application of this method, in section 4 we present a sufficient condition for proving the existence of global-in-time solutions. As a result, the existence of a global-in-time solution to a certain semilinear parabolic equation is proved.

2. Verification theory in a time interval.

2.1. Preliminaries. Throughout this paper, the space of all pth power Lebesgue integrable functions on Ω is denoted by $L^p(\Omega)$ for $p \in [1,\infty)$ and $L^\infty(\Omega) := \{v : \operatorname{ess\,sup}_{x\in\Omega} |v(x)| < \infty\}$. For $p = 2$, the inner product on $L^2(\Omega)$ is defined by

$$(v, w)_{L^2} := \int_\Omega v(x)w(x)dx.$$

For a positive integer m and $p \in [1,\infty]$, let $W^{m,p}(\Omega)$ be the mth order Sobolev space of $L^p(\Omega)$. Let $H^m(\Omega) := W^{m,2}(\Omega)$ and $H_0^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$, where the condition that $v = 0$ on $\partial\Omega$ is meant in the trace sense. Furthermore, $H^{-1}(\Omega)$ denotes the topological dual space of $H_0^1(\Omega)$, i.e., the space of linear continuous functionals in $H_0^1(\Omega)$. We employ the usual norms, given by

$$\|v\|_{L^2} := \sqrt{(v,v)_{L^2}}, \quad \|v\|_{H_0^1} := \|\nabla v\|_{L^2}, \quad \text{and } \|\varphi\|_{H^{-1}} := \sup_{\substack{0\ne v\in H_0^1(\Omega)\\ \|v\|_{H_0^1}=1}} |\langle\varphi, v\rangle|,$$

where $\langle\cdot,\cdot\rangle$ is a dual product between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

Let $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be a bilinear form that is defined by

$$a(v, w) := \sum_{1\le i,j\le 2} \left(a_{ij}(x)\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_j}\right)_{L^2}.$$

Then, for $\phi \in D(A)$, it holds that $a(\phi, v) = (A\phi, v)_{L^2}$ for all $v \in H_0^1(\Omega)$. From the assumptions on A, it follows that the bilinear form $a(\cdot,\cdot)$ satisfies the condition of strong coercivity,

$$a(v, v) \ge \mu\|v\|_{H_0^1}^2, \quad \forall v \in H_0^1(\Omega), \tag{3}$$

and the condition of boundedness, meaning that there exists $M > 0$ such that

$$|a(v,w)| \le M\|v\|_{H_0^1}\|w\|_{H_0^1} \quad \forall v,w \in H_0^1(\Omega). \tag{4}$$

We define $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ as

$$\langle \mathcal{A}v, w\rangle := a(v,w) \quad \forall w \in H_0^1(\Omega). \tag{5}$$

Let $\{\psi_i\}_{i\in\mathbb{N}}$ be a complete orthonormal system of eigenfunctions; namely, each $\psi_i \in H_0^1(\Omega)$ is an eigenfunction of $\mathcal{A}$ satisfying $\langle \psi_i, \psi_j\rangle = \delta_{i,j}$, where $\delta_{i,j}$ is Kronecker's delta. The spectrum[4] of $\mathcal{A}$ is denoted by $\sigma(\mathcal{A})$. For $\alpha \in (0,1)$, a fractional power of $\mathcal{A}$ is defined by

$$\mathcal{A}^\alpha\phi := \sum_{j=1}^{\infty} \lambda_j^\alpha c_j\psi_j, \quad D(\mathcal{A}^\alpha) := \left\{\phi = \sum_{j=1}^{\infty} c_j\psi_j \in H^{-1}(\Omega) \ : \ \sum_{j=1}^{\infty} c_j^2\lambda_j^\alpha < \infty\right\},$$

where $c_i = \langle \phi, \psi_i\rangle$ and $\{\lambda_i\}_{i\in\mathbb{N}} = \sigma(\mathcal{A})$. The following lemma holds for $\mathcal{A}^{\frac{1}{2}}$.

LEMMA 2.1. *The relation $D(\mathcal{A}^{\frac{1}{2}}) = L^2(\Omega)$ holds. Moreover,*

$$\mu^{\frac{1}{2}}\|w\|_{L^2} \le \|\mathcal{A}^{\frac{1}{2}}w\|_{H^{-1}} \le M^{\frac{1}{2}}\|w\|_{L^2} \tag{6}$$

is satisfied for all $w \in L^2(\Omega)$.

The proof of Lemma 2.1 is given in [29]. In the following, we present a fundamental theorem for the analytic semigroup $\{e^{-t\mathcal{A}}\}_{t\ge 0}$ generated by $-\mathcal{A}$. Proofs of this theorem can be found in several textbooks, e.g., [23].

THEOREM 2.2. *Let $x \in H_0^1(\Omega)$, and let λ_0 be a positive number. Assume that $\mathcal{A}$ satisfies*

$$\langle -\mathcal{A}x, x\rangle \le 0, \quad R(\lambda_0 I + \mathcal{A}) = H^{-1}(\Omega),$$

where $R(\mathcal{B})$ denotes the range of any operator $\mathcal{B}$. Then, there exists an analytic semigroup $\{e^{-t\mathcal{A}}\}_{t\ge 0}$ generated by $-\mathcal{A}$.

Using Theorem 2.2, it follows from (3) and (4) that the operator $-\mathcal{A}$ generates an analytic semigroup $\{e^{-t\mathcal{A}}\}_{t\ge 0}$ over $H^{-1}(\Omega)$. In this paper, the function $u \in L^\infty(J; H_0^1(\Omega))$ given by

$$u(t) = e^{-(t-t_0)\mathcal{A}}u_0 + \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} f(u(s))\, ds$$

is called a mild solution of (1).

Now, we provide two lemmas concerning the relationship between fractional powers of $\mathcal{A}$ and the analytic semigroup $e^{-t\mathcal{A}}$.

LEMMA 2.3. *For $\varphi \in H^{-1}(\Omega)$, it follows that*

$$\partial_t e^{-t\mathcal{A}}\varphi = -\mathcal{A}e^{-t\mathcal{A}}\varphi.$$

Moreover, if $\phi \in D(\mathcal{A}^\alpha)$, the following holds for $\alpha \in (0,1]$:

$$-\mathcal{A}^\alpha e^{-t\mathcal{A}}\phi = -e^{-t\mathcal{A}}\mathcal{A}^\alpha\phi.$$

[4]As the inverse of the operator A is a compact self-adjoint operator, the spectral theorem implies that the operator $\mathcal{A}$ admits positive discrete spectra (see, e.g., [3]).

On the basis of the representation formula given by the Dunford integral, the spectral mapping theorem can be applied to $e^{-t\mathcal{A}}$. Thus, we can derive the following lemma.

LEMMA 2.4. *Let $\alpha \in (0,1)$. Then, for a fixed $\beta \in (0,1)$ it holds that*

$$\left\|\mathcal{A}^{\alpha} e^{-t\mathcal{A}}\varphi\right\|_{H^{-1}} \le \left(\frac{\alpha}{e\beta t}\right)^{\alpha} e^{-(1-\beta)t\lambda_{\min}}\|\varphi\|_{H^{-1}} \quad \forall \varphi \in H^{-1}(\Omega), \tag{7}$$

where $\lambda_{\min}$ is the minimal eigenvalue of $\mathcal{A}$ and e is the Euler number.

Proof. Because the minimum value of $\sigma(\mathcal{A})$ is positive, we have

$$\sup_{x\in\sigma(\mathcal{A})} \left|x^{\alpha} e^{-\beta t x}\right| \le \left(\frac{\alpha}{e\beta t}\right)^{\alpha} \quad \text{and} \quad \sup_{x\in\sigma(\mathcal{A})} \left|e^{-(1-\beta)tx}\right| \le e^{-(1-\beta)t\lambda_{\min}}.$$

Therefore, the spectral mapping theorem implies that the following inequality holds:

$$\begin{aligned}\left\|\mathcal{A}^{\alpha} e^{-t\mathcal{A}}\varphi\right\|_{H^{-1}} &= \sup_{x\in\sigma(\mathcal{A})} \left|x^{\alpha} e^{-tx}\right| \|\varphi\|_{H^{-1}} \\ &\le \sup_{x\in\sigma(\mathcal{A})} \left|x^{\alpha} e^{-\beta t x}\right| \sup_{x\in\sigma(\mathcal{A})} \left|e^{-(1-\beta)tx}\right| \|\varphi\|_{H^{-1}}\end{aligned}$$

for all $\varphi \in H^{-1}(\Omega)$. This in turn implies (7). ∎

From Lemma 2.4, it follows that if we set $\alpha = \beta = 1/2$, the following corollary holds.

COROLLARY 2.5.

$$\left\|\mathcal{A}^{\frac{1}{2}} e^{-t\mathcal{A}}\varphi\right\|_{H^{-1}} \le e^{-\frac{1}{2}} t^{-\frac{1}{2}} e^{\frac{-t\lambda_{\min}}{2}} \|\varphi\|_{H^{-1}} \quad \forall \varphi \in H^{-1}(\Omega).$$

We remark that the verified inclusion of $\lambda_{\min}$ can be obtained by a natural extension of the method described in [17]. The upper bound can easily be obtained using the Rayleigh–Ritz method for a finite-dimensional space V_h satisfying $V_h \subset H_0^1(\Omega)$. For the lower bound, we apply the main theorem of the paper [17], which we state as follows.

THEOREM 2.6. *Let V_h be an N-dimensional subspace of $H_0^1(\Omega)$, and let $R_h : H_0^1(\Omega) \to V_h$ be the Ritz projection defined by $a(v - R_h v, v_h) = 0$ for all $v_h \in V_h$. Assume that there exists a computable error estimate; that is, we can determine $C_M > 0$ such that*

$$\|v - R_h v\|_{L^2} \le C_M \|v - R_h v\|_{H_0^1} \quad \textit{for } v \in H_0^1(\Omega).$$

Furthermore, let $\lambda_{h,k}$ be the kth eigenvalue of the eigenvalue problem over V_h; find $v_h \in V_h$ and $\lambda_h \in \mathbb{R}$ such that $a(v_h, w_h) = \lambda_h (v_h, w_h)_{L^2}$ for all $w_h \in V_h$. Then, a lower bound of λ_k is given by

$$\frac{\lambda_{h,k}}{1 + C_M^2 \lambda_{h,k}} \le \lambda_k \quad (k = 1, 2, \dots, N).$$

2.2. Verification theorem. Let $J = (t_0, t_1]$ and $\tau := t_1 - t_0$, where $t_0, t_1 \in \mathbb{R}$ such that $0 \le t_0 < t_1 < \infty$. In this subsection, we present a theorem that gives a sufficient condition for the existence and local uniqueness of a mild solution for (1).

THEOREM 2.7. *Let us consider the initial-boundary value problem defined by* (1), *and let* V_h *be a finite-dimensional subspace of* $H_0^1(\Omega)$. *For* $\hat{u}_0, \hat{u}_1 \in V_h \cap L^\infty(\Omega)$, *we define* $\omega(t)$ *as*

$$\omega(t) = \hat{u}_0\phi_0(t) + \hat{u}_1\phi_1(t), \quad t \in J,$$

where $\phi_i(t)$ $(i = 0, 1)$ *is a linear Lagrange basis satisfying* $\phi_i(t_j) = \delta_{ij}$ $(\delta_{ij}$ *is the Kronecker delta with* $j = 0, 1)$.

Assume that the initial function u_0 *satisfies* $\|u_0 - \hat{u}_0\|_{H_0^1} \le \varepsilon_0$, *and that* ω *satisfies the following estimate:*

$$\operatorname*{ess\,sup}_{t\in J} \left\| \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} (\partial_t \omega(s) + \mathcal{A}\omega(s) - f(\omega(s)))ds \right\|_{H_0^1} \le \delta. \tag{8}$$

Furthermore, assume that f *satisfies*

$$\|f(\varphi) - f(\psi)\|_{L^\infty(J;L^2(\Omega))} \le L_{\rho_0} \|\varphi - \psi\|_{L^\infty(J;H_0^1(\Omega))} \quad \forall \varphi, \psi \in B_J(\omega, \rho_0), \tag{9}$$

for all $\rho_0 \in (0, \rho]$ *for a certain* $\rho > 0$. *If*

$$\frac{M}{\mu}\varepsilon_0 + \frac{2}{\mu}\sqrt{\frac{M\tau}{e}} L_\rho \rho + \delta < \rho,$$

then a mild solution $u(t)$ *of* (1) *uniquely exists in the ball* $B_J(\omega, \rho)$, *where* $t \in J$.

Remark 2.8. In (8) the integrand $e^{-(t-s)\mathcal{A}}(\partial_s\omega(s) + \mathcal{A}\omega(s) - f(\omega(s))) \in H_0^1(\Omega)$ for $s \in J$, because $e^{-(t-s)\mathcal{A}}$ is an analytic semigroup. It is assumed in this theorem that the integrand is bounded and integrable in the sense of Bochner.

Proof. Let us define an operator $S : L^\infty(J; H_0^1(\Omega)) \to L^\infty(J; H^{-1}(\Omega))$ using the analytic semigroup $e^{-t\mathcal{A}}$ as

$$(S(z))(t) := e^{-(t-t_0)\mathcal{A}}(u_0 - \hat{u}_0) + \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} g(z(s))ds, \tag{10}$$

where $g(z(t)) = f(\omega(t)+z(t))-(\partial_t\omega(t)+\mathcal{A}\omega(t))$. In (10), the integrand $e^{-(t-s)\mathcal{A}}g(z(s))$ is in $H_0^1(\Omega)$ in the sense of Bochner for $s \in J$, because $e^{-(t-s)\mathcal{A}}$ is an analytic semigroup. The fact that the integral is bounded follows by assumptions (8) and (9).

For $\rho > 0$, let $Z := \{z \in L^\infty(J; H_0^1(\Omega)) : \|z\|_{L^\infty(J;H_0^1(\Omega))} \le \rho\}$. We note that u is a mild solution if and only if $z := u - \omega$ is a fixed point of the operator S in Z. In the following, on the basis of Banach's fixed-point theorem, we derive a sufficient condition of S having a fixed point in Z. First, we derive a condition guaranteeing that $S(Z) \subset Z$.

We will employ the inequality

$$\mu\|\phi\|_{H_0^1} \le \|\mathcal{A}\phi\|_{H^{-1}} \le M\|\phi\|_{H_0^1} \quad \forall \phi \in H_0^1(\Omega), \tag{11}$$

which follows from (3)–(5).

The first term in the right-hand side of (10) can be estimated, using the spectral mapping theorem and (11), as

$$\begin{aligned} \left\| e^{-(t-t_0)\mathcal{A}}(u_0 - \hat{u}_0) \right\|_{H_0^1} &\le \mu^{-1} \left\| \mathcal{A}\, e^{-(t-t_0)\mathcal{A}}(u_0 - \hat{u}_0) \right\|_{H^{-1}} \\ &\le \frac{M}{\mu} e^{-(t-t_0)\lambda_{\min}} \varepsilon_0, \end{aligned} \tag{12}$$

where $\lambda_{\min}$ denotes the minimal eigenvalue of $\mathcal{A}$ in $H^{-1}(\Omega)$. Therefore, it follows that

$$\left\| e^{-(t-t_0)\mathcal{A}}(u_0 - \hat{u}_0) \right\|_{L^\infty(J;H_0^1(\Omega))} \le \frac{M}{\mu}\varepsilon_0. \tag{13}$$

To estimate the second term in the right-hand side of (10), we decompose $g(z(s)) \in H^{-1}(\Omega)$ for $z \in Z$ into two parts as

$$\begin{aligned} g(z(s)) &= f(\omega(s) + z(s)) - (\partial_t \omega(s) + \mathcal{A}\omega(s)) \\ &= g_1(s) + g_2(s), \end{aligned}$$

where

$$g_1(s) := f(\omega(s) + z(s)) - f(\omega(s))$$

and

$$g_2(s) := f(\omega(s)) - (\partial_t \omega(s) + \mathcal{A}\omega(s)),$$

respectively. Now, set

$$\nu(t) := \int_{t_0}^{t} (t-s)^{-\frac{1}{2}} e^{-\frac{1}{2}(t-s)\lambda_{\min}} ds,$$

so that we have

$$\sup_{t \in J} \nu(t) \le \sup_{t \in J} \int_{t_0}^{t} (t-s)^{-\frac{1}{2}} ds = 2\sqrt{\tau}. \tag{14}$$

Then, using (6) in Lemma 2.1, (11), and Corollary 2.5, it follows that

$$\begin{aligned} &\left\| \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} g_1(s) ds \right\|_{H_0^1} \\ &= \left\| \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} (f(\omega(s) + z(s)) - f(\omega(s))) ds \right\|_{H_0^1} \\ &\le \mu^{-1} \int_{t_0}^{t} \left\| \mathcal{A}\, e^{-(t-s)\mathcal{A}} (f(\omega(s) + z(s)) - f(\omega(s))) \right\|_{H^{-1}} ds \\ &= \mu^{-1} \int_{t_0}^{t} \left\| \mathcal{A}^{\frac{1}{2}}\, e^{-(t-s)\mathcal{A}} \mathcal{A}^{\frac{1}{2}} (f(\omega(s) + z(s)) - f(\omega(s))) \right\|_{H^{-1}} ds \\ &\le \mu^{-1} e^{-\frac{1}{2}} \int_{t_0}^{t} (t-s)^{-\frac{1}{2}} e^{-\frac{1}{2}(t-s)\lambda_{\min}} \left\| \mathcal{A}^{\frac{1}{2}} (f(\omega(s) + z(s)) - f(\omega(s))) \right\|_{H^{-1}} ds \\ &\le \mu^{-1} M^{\frac{1}{2}} e^{-\frac{1}{2}} \int_{t_0}^{t} (t-s)^{-\frac{1}{2}} e^{-\frac{1}{2}(t-s)\lambda_{\min}} \left\| f(\omega(s) + z(s)) - f(\omega(s)) \right\|_{L^2} ds \\ &\le \mu^{-1} M^{\frac{1}{2}} e^{-\frac{1}{2}} \nu(t) \left\| f(\omega + z) - f(\omega) \right\|_{L^\infty(J;L^2(\Omega))}. \end{aligned}$$

Furthermore, (9) and (14) yield that

$$\left\| \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} g_1(s) ds \right\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \le \frac{2}{\mu}\sqrt{\frac{M\tau}{e}} L_\rho \rho. \tag{15}$$

From (8) we have

$$
\begin{aligned}
(16) \qquad & \left\| \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} g_2(s) ds \right\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \\
&= \left\| \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} (\partial_t \omega(s) + \mathcal{A}\omega(s) - f(\omega(s))) ds \right\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \le \delta.
\end{aligned}
$$

Now, it follows from (13), (15), and (16) that

$$
\|S(z)\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \le \frac{M}{\mu}\varepsilon_0 + \frac{2}{\mu}\sqrt{\frac{M\tau}{e}} L_\rho \rho + \delta.
$$

Then, the fact that the desired estimate $\|S(z)\|_{L^\infty(J;H_0^1(\Omega))} < \rho$ holds follows from the conditions stated in the theorem, and this implies that $S(z) \in Z$.

Next, we show that S is a contraction mapping on Z under the assumption of the theorem. For any z_1 and z_2 in Z, it holds that

$$
S(z_1) - S(z_2) = \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} \left\{ f(z_1(s) + \omega(s)) - f(z_2(s) + \omega(s)) \right\} ds.
$$

Then, we have

$$
\begin{aligned}
& \left\| \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} \left\{ f(z_1(s) + \omega(s)) - f(z_2(s) + \omega(s)) \right\} ds \right\|_{H_0^1} \\
& \le \mu^{-1} M^{\frac{1}{2}} e^{-\frac{1}{2}} \nu(t) \|f(z_1 + \omega) - f(z_2 + \omega)\|_{L^\infty(J;L^2(\Omega))}.
\end{aligned}
$$

Here, it holds that $z_i + \omega \in B_J(\omega, \rho)$ $(i = 1, 2)$. From (9) and (14), we obtain

$$
\|S(z_1) - S(z_2)\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \le \frac{2}{\mu}\sqrt{\frac{M\tau}{e}} L_\rho \|z_1 - z_2\|_{L^\infty\left(J;H_0^1(\Omega)\right)}.
$$

Furthermore, the conditions of the theorem imply that

$$
\frac{2}{\mu}\sqrt{\frac{M\tau}{e}} L_\rho < 1.
$$

Therefore, S becomes a contraction mapping, and Banach's fixed-point theorem asserts that there exists a unique fixed point of S in Z. ∎

Because S has a fixed point in Z, the following a posteriori error estimate at $t = t_1$ holds.

THEOREM 2.9. *Assume that the conditions of Theorem* 2.7 *are satisfied, so that the existence and local uniqueness of a mild solution $u(t)$ is guaranteed in $B_J(\omega, \rho)$ for $t \in J$. Furthermore, assume also that ω satisfies*

$$
(17) \qquad \left\| \int_{t_0}^{t_1} e^{-(t_1-s)\mathcal{A}} \left(\partial_t \omega(s) + \mathcal{A}\omega(s) - f(\omega(s)) \right) ds \right\|_{H_0^1} \le \tilde{\delta}.
$$

Then, the following a posteriori error estimate holds:

$$
\|u(t_1) - \hat{u}_1\|_{H_0^1} \le \frac{M}{\mu} e^{-\tau\lambda_{\min}} \varepsilon_0 + \frac{2}{\mu}\sqrt{\frac{M\tau}{e}} L_\rho \rho + \tilde{\delta} =: \varepsilon_1.
$$

Proof. Because $z = u - \omega$ is a fixed point of S, it holds that $\|z(t_1)\|_{H_0^1} = \|S(z(t_1))\|_{H_0^1}$. By setting $t = t_1$ in (12), (15), and (17), we obtain the following estimate:

$$\|S(z(t_1))\|_{H_0^1} \le \frac{M}{\mu} e^{-\tau\lambda_{\min}} \varepsilon_0 + \frac{2}{\mu}\sqrt{\frac{M\tau}{e}} L_\rho \rho + \tilde{\delta}. \tag{18}$$ ∎

2.3. Residual estimation. This subsection is devoted to presenting a method of calculating residual estimates δ in (8) and $\tilde{\delta}$ in (17). For $\hat{u}_1, \hat{u}_0 \in V_h \cap L^\infty(\Omega)$, we define $\mathcal{B}(\hat{u}_1) \in H^{-1}(\Omega)$ as

$$\langle \mathcal{B}(\hat{u}_1), v\rangle := \left(\frac{\hat{u}_1 - \hat{u}_0}{\tau}, v\right)_{L^2} + a(\hat{u}_1, v) - (f(\hat{u}_1), v)_{L^2} \quad \forall v \in H_0^1(\Omega),$$

and $\mathcal{F}(\hat{u}_1) \in H^{-1}(\Omega)$ as

$$\langle \mathcal{F}(\hat{u}_1), v\rangle := \left(\frac{\hat{u}_1 - \hat{u}_0}{\tau}, v\right)_{L^2} + a(\hat{u}_0, v) - (f(\hat{u}_0), v)_{L^2} \quad \forall v \in H_0^1(\Omega).$$

By applying techniques of verified numerical computations for operator equations (e.g., [27]), we can numerically evaluate $\beta, \eta > 0$, such that

$$\|\mathcal{B}(\hat{u}_1)\|_{H^{-1}} \le \beta \quad \text{and} \quad \|\mathcal{B}(\hat{u}_1) - \mathcal{F}(\hat{u}_1)\|_{H^{-1}} \le \eta.$$

Let $p(t) := f(\hat{u}_1)\phi_1(t) + f(\hat{u}_0)\phi_0(t)$. Then, the function $p(t)$ is a linear approximation of $f(\omega(t))$. The classical error bound of the linear interpolation yields that for a fixed $x \in \Omega$,

$$|f(\omega(t)) - p(t)| \le \frac{\tau^2}{8} \sup_{t \in J} \left| \frac{d^2}{dt^2} f(\omega(t)) \right|.$$

It follows that

$$\frac{d^2}{dt^2} f(\omega(t)) = f''[\omega(t)] \left(\frac{d}{dt}\omega(t)\right)^2 + f'[\omega]\left(\frac{d^2}{dt^2}\omega(t)\right).$$

The second term in the right-hand side vanishes, because ω is a linear function corresponding to t. Then, we obtain that

$$\begin{aligned} |f(\omega(t)) - p(t)| &\le \frac{\tau^2}{8} \sup_{t \in J} \left| f''[\omega(t)] \left(\frac{d}{dt}\omega(t)\right)^2 \right| \\ &\le \frac{1}{8}\left(\sup_{t \in J} |f''[\omega(t)]|\right) \|\hat{u}_1 - \hat{u}_0\|_{L^\infty}^2, \end{aligned}$$

where $f''[\omega(t)]$ is the second order Fréchet derivative of f at $\omega(t)$. Because $\|v\|_{L^2} \le |\Omega|^{\frac{1}{2}} \|v\|_{L^\infty}$ holds for all $v \in L^\infty(\Omega)$, it follows that, for $t \in J$,

$$\|f(\omega(t)) - p(t)\|_{L^2} \le \frac{|\Omega|^{\frac{1}{2}}}{8} \left(\sup_{t \in J,\, x \in \Omega} |f''[\omega(t)]|\right) \|\hat{u}_1 - \hat{u}_0\|_{L^\infty}^2. \tag{19}$$

From the facts that $\omega(t) = \hat{u}_1\phi_1(t) + \hat{u}_0\phi_0(t)$ and $\phi_1(t) + \phi_0(t) = 1$, it follows that

$$\begin{aligned}
f(\omega(t)) - (\partial_t\omega(t) + \mathcal{A}\omega(t)) &= f(\omega(t)) - p(t) + p(t) - \left(\frac{\hat{u}_1 - \hat{u}_0}{\tau} + \mathcal{A}\omega(t)\right) \\
&= f(\omega(t)) - p(t) + \left(f(\hat{u}_1) - \frac{\hat{u}_1 - \hat{u}_0}{\tau} - \mathcal{A}\hat{u}_1\right)\phi_1(t) \\
&\quad + \left(f(\hat{u}_0) - \frac{\hat{u}_1 - \hat{u}_0}{\tau} - \mathcal{A}\hat{u}_0\right)\phi_0(t) \\
&= f(\omega(t)) - p(t) - (\mathcal{B}(\hat{u}_1)\phi_1(t) + \mathcal{F}(\hat{u}_1)\phi_0(t)).
\end{aligned}$$

Let $\|\hat{u}_1 - \hat{u}_0\|_{L^\infty} \le \alpha$. Here, we know from (19) that

$$\begin{aligned}
&\left\|\int_{t_0}^t e^{-(t-s)\mathcal{A}}(f(\omega(s)) - p(s))ds\right\|_{H_0^1} \\
&\le \mu^{-1}\int_{t_0}^t \left\|\mathcal{A}^{\frac{1}{2}}\, e^{-(t-s)\mathcal{A}}\mathcal{A}^{\frac{1}{2}}(f(\omega(s)) - p(s))\right\|_{H^{-1}} ds \\
&\le \mu^{-1}e^{-\frac{1}{2}}\int_{t_0}^t (t-s)^{-\frac{1}{2}}e^{-\frac{1}{2}(t-s)\lambda_{\min}}\left\|\mathcal{A}^{\frac{1}{2}}(f(\omega(s)) - p(s))\right\|_{H^{-1}} ds \\
&\le \mu^{-1}M^{\frac{1}{2}}e^{-\frac{1}{2}}\int_{t_0}^t (t-s)^{-\frac{1}{2}}e^{-\frac{1}{2}(t-s)\lambda_{\min}}\|f(\omega(s)) - p(s)\|_{L^2}\, ds \\
&\le \mu^{-1}M^{\frac{1}{2}}e^{-\frac{1}{2}}\nu(t)\frac{|\Omega|^{\frac{1}{2}}}{8}C_\omega\alpha^2,
\end{aligned}$$

where $C_\omega := \max_{t\in J,\, x\in\Omega}|f''[\omega(t)]|$. Combining this with (14) yields that

$$\left\|\int_{t_0}^t e^{-(t-s)\mathcal{A}}(f(\omega(s)) - p(s))ds\right\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \le \frac{|\Omega|^{\frac{1}{2}}}{4\mu}\sqrt{\frac{M\tau}{e}}C_\omega\alpha^2. \tag{20}$$

Next, we can perform an integration by parts to obtain that

$$\begin{aligned}
&\int_{t_0}^t \partial_s e^{-(t-s)\mathcal{A}}(\mathcal{B}(\hat{u}_1)\phi_1(s) + \mathcal{F}(\hat{u}_1)\phi_0(s))ds \\
&= \mathcal{B}(\hat{u}_1)\phi_1(t) + \left(\phi_0(t) - e^{-(t-t_0)\mathcal{A}}\right)\mathcal{F}(\hat{u}_1) - \tau^{-1}\int_{t_0}^t e^{-(t-s)\mathcal{A}}(\mathcal{B}(\hat{u}_1) - \mathcal{F}(\hat{u}_1))ds.
\end{aligned}$$

Because $\mathcal{A}\, e^{-(t-s)\mathcal{A}} = \partial_s e^{-(t-s)\mathcal{A}}$, we can consider (11) and the spectral mapping theorem, to yield

$$\begin{aligned}
&\left\|\int_{t_0}^t e^{-(t-s)\mathcal{A}}(\mathcal{B}(\hat{u}_1)\phi_1(s) + \mathcal{F}(\hat{u}_1)\phi_0(s))ds\right\|_{H_0^1} \qquad (21)\\
&\le \mu^{-1}\left\|\int_{t_0}^t \partial_s e^{-(t-s)\mathcal{A}}(\mathcal{B}(\hat{u}_1)\phi_1(s) + \mathcal{F}(\hat{u}_1)\phi_0(s))ds\right\|_{H^{-1}} \\
&\le \mu^{-1}\Bigg(\left\|\mathcal{B}(\hat{u}_1)\phi_1(t) + \left(\phi_0(t) - e^{-(t-t_0)\mathcal{A}}\right)\mathcal{F}(\hat{u}_1)\right\|_{H^{-1}} \\
&\qquad + \tau^{-1}\int_{t_0}^t \left\|e^{-(t-s)\mathcal{A}}(\mathcal{B}(\hat{u}_1) - \mathcal{F}(\hat{u}_1))\right\|_{H^{-1}} ds\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\le \mu^{-1}\Bigg(\left\| (\mathcal{B}(\hat{u}_1) - \mathcal{F}(\hat{u}_1))\, \phi_1(t) + \left(I - e^{-(t-t_0)\mathcal{A}} \right) \mathcal{F}(\hat{u}_1) \right\|_{H^{-1}} \\
&\qquad\qquad + \tau^{-1} \int_{t_0}^{t} \left\| e^{-(t-s)\mathcal{A}} (\mathcal{B}(\hat{u}_1) - \mathcal{F}(\hat{u}_1)) \right\|_{H^{-1}} ds \Bigg) \\
&= \mu^{-1}\Bigg(\left\| \left(\phi_1(t) - \left(I - e^{-(t-t_0)\mathcal{A}} \right) \right) (\mathcal{B}(\hat{u}_1) - \mathcal{F}(\hat{u}_1)) + \left(I - e^{-(t-t_0)\mathcal{A}} \right) \mathcal{B}(\hat{u}_1) \right\|_{H^{-1}} \\
&\qquad\qquad + \tau^{-1} \int_{t_0}^{t} \left\| e^{-(t-s)\mathcal{A}} (\mathcal{B}(\hat{u}_1) - \mathcal{F}(\hat{u}_1)) \right\|_{H^{-1}} ds \Bigg) \\
&\le \mu^{-1}\left\{ \|\mathcal{B}(\hat{u}_1)\|_{H^{-1}} + \left(\phi_1(t) + 1 + \frac{1 - e^{-(t-t_0)\lambda_{\min}}}{\tau \lambda_{\min}} \right) \|\mathcal{B}(\hat{u}_1) - \mathcal{F}(\hat{u}_1)\|_{H^{-1}} \right\}.
\end{aligned}
$$

Then, because $\|\mathcal{B}(\hat{u}_1)\|_{H^{-1}} \le \beta$ and $\|\mathcal{B}(\hat{u}_1) - \mathcal{F}(\hat{u}_1)\|_{H^{-1}} \le \eta$, it follows that

$$
\begin{aligned}
(22) \qquad & \left\| \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} (\mathcal{B}(\hat{u}_1)\phi_1(s) + \mathcal{F}(\hat{u}_1)\phi_0(s)) ds \right\|_{L^\infty\left(J; H_0^1(\Omega)\right)} \\
&\le \mu^{-1} \left\{ \beta + \left(2 + \frac{1 - e^{-\tau\lambda_{\min}}}{\tau\lambda_{\min}} \right) \eta \right\}.
\end{aligned}
$$

Finally, from (20) and (22) we have

$$
\begin{aligned}
(23) \qquad & \left\| \int_{t_0}^{t} e^{-(t-s)\mathcal{A}} (\partial_t \omega(s) + \mathcal{A}\omega(s) - f(\omega(s))) ds \right\|_{L^\infty\left(J; H_0^1(\Omega)\right)} \\
&\le \frac{|\Omega|^{\frac{1}{2}}}{4\mu} \sqrt{\frac{M\tau}{e}} C_\omega \alpha^2 + \mu^{-1} \left\{ \beta + \left(2 + \frac{1 - e^{-\tau\lambda_{\min}}}{\tau\lambda_{\min}} \right) \eta \right\}.
\end{aligned}
$$

Then, we can choose the right-hand side of (23) as δ.

By setting $t = t_1$ in (21), we obtain the same estimate as in (23). Thus, the right-hand side of (23) can also be taken as $\tilde{\delta}$.

2.4. Local Lipschitz bound of f. In this subsection, we provide a formula for calculating the local Lipschitz bound L_ρ when $f(u) = c_1 u + c_2 u^2 + c_3 u^3$ for $c_i \in \mathbb{R}$ $(i = 1, 2, 3)$. The Fréchet derivative of $f : H_0^1(\Omega) \to L^2(\Omega)$ at $y \in B_J(\omega, \rho)$ is given by $f'[y] = c_1 + 2c_2 y + 3c_3 y^2$. Hölder's inequality gives the following estimate for all $\phi \in H_0^1(\Omega)$:

$$
\begin{aligned}
&\|f'[y]\phi\|_{L^\infty(J; L^2(\Omega))} \\
&= \sup_{t\in J} \left\| \left(c_1 + 2c_2 y(t) + 3c_3 y(t)^2 \right) \phi \right\|_{L^2} \\
&\le \sup_{t\in J} \left(|c_1| \|\phi\|_{L^2} + 2|c_2| \|y(t)\|_{L^4} \|\phi\|_{L^4} + 3|c_3| \|y(t)\|_{L^6}^2 \|\phi\|_{L^6} \right) \\
&\le \sup_{t\in J} \left\{ \left(|c_1| C_2 + 2|c_2| C_4^2 \|y(t)\|_{H_0^1} + 3|c_3| C_6^3 \|y(t)\|_{H_0^1}^2 \right) \|\phi\|_{H_0^1} \right\} \\
&\le \left\{ |c_1| C_2 + 2|c_2| C_4^2 \left(\|\omega\|_{L^\infty\left(J; H_0^1(\Omega)\right)} + \rho \right) + 3|c_3| C_6^3 \left(\|\omega\|_{L^\infty\left(J; H_0^1(\Omega)\right)} + \rho \right)^2 \right\} \|\phi\|_{H_0^1},
\end{aligned}
$$

where C_p $(p = 2, 4, 6)$ is the Sobolev embedding constant[5] satisfying $\|\phi\|_{L^p} \le C_p \|\phi\|_{H_0^1}$ for all $\phi \in H_0^1(\Omega)$. Thus, the local Lipschitz bound L_ρ can be taken as

$$L_\rho = |c_1| C_2 + 2|c_2| C_4^2 \left(\|\omega\|_{L^\infty\left(J; H_0^1(\Omega)\right)} + \rho \right) + 3|c_3| C_6^3 \left(\|\omega\|_{L^\infty\left(J; H_0^1(\Omega)\right)} + \rho \right)^2 .$$

Remark 2.10. The operator "$-A$" also generates an analytic semigroup $\{e^{-tA}\}_{t \ge 0}$ over $L^2(\Omega)$. The main theorem (Theorem 2.7) could also be constructed using the analytic semigroup $\{e^{-tA}\}_{t \ge 0}$. Such a result is given in the paper [18]. The main difference between the result in [18] and the one in this paper is the residual estimate. In [18], letting

$$\mathcal{C}_1 := \frac{\hat{u}_1 - \hat{u}_0}{\tau} + A\hat{u}_1 - f(\hat{u}_1) \quad \text{and} \quad \mathcal{C}_0 := \frac{\hat{u}_1 - \hat{u}_0}{\tau} + A\hat{u}_0 - f(\hat{u}_0),$$

the residual estimate is given by

$$\begin{aligned} &\left\| \int_{t_0}^{t} A^{1/2} e^{-(t-s)A} (\partial_t \omega(s) + A\omega(s) - f(\omega(s))) ds \right\|_{L^\infty(J; L^2(\Omega))} \\ &\le \sqrt{\frac{2\pi}{\lambda_{\min} e}} \operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}} \right) \left(\frac{C_4^2 C_\omega}{8} \alpha^2 + \|\mathcal{C}_1\|_{L^2} + \|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2} \right). \end{aligned}$$

This estimate is sharper than (23), but the additional assumption that $V_h \subset D(A)$ is required.

3. Proof of existence on several time intervals.

3.1. Concatenation scheme of verified numerical inclusion. For a fixed natural number n, let $0 \le t_0 < t_1 < \cdots < t_n < \infty$, $J_k = (t_{k-1}, t_k]$, $\tau_k := t_k - t_{k-1}$ $(k = 1, 2, \ldots, n)$, and $J = \cup J_k$. In this subsection, we will demonstrate a recursive scheme for proving the existence and local uniqueness of a mild solution for the initial-boundary value problem (1).

For $i = 0, 1, \ldots, n$, let $\hat{u}_i \in V_h$. Then, we define

$$\omega(t) := \sum_{i=0}^{n} \hat{u}_i \phi_i(t), \quad t \in J,$$

where $\phi_i(t)$ is a piecewise linear Lagrange basis satisfying $\phi_i(t_j) = \delta_{ij}$ (δ_{ij} is the Kronecker delta with $j = 0, 1, \ldots, n$).

Because u_0 is a given function, we can calculate some ε_0 that satisfies $\|u_0 - \hat{u}_0\|_{H_0^1} \le \varepsilon_0$. Then, on the basis of Theorem 2.7 with $J = J_1$, we attempt to enclose the mild solution $u(t)$, with $t \in J_1$. If the sufficient condition of Theorem 2.7 is satisfied, then the mild solution is enclosed in a ball centered at $\omega(t)$, $t \in J_1$, with a radius of ρ_1. Furthermore, we have an a posteriori error estimate based on Theorem 2.9, given by $\|u(t_1) - \hat{u}_1\|_{H_0^1} \le \varepsilon_1$. From the error estimate $\|u(t_1) - \hat{u}_1\|_{H_0^1} \le \varepsilon_1$, we attempt to further enclose the mild solution $u(t)$ on the basis of Theorem 2.7, with $J = J_2$. We repeat this process recursively. That is, using the error estimate $\|u(t_{k-1}) - \hat{u}_{k-1}\|_{H_0^1} \le \varepsilon_{k-1}$ $(k = 1, 2, \ldots, n)$, we recursively prove the existence and local uniqueness of the mild solution in each ball

$$B_{J_k}(\omega|_{t \in J_k}, \rho_k) = \left\{ y \in L^\infty\left(J_k; H_0^1(\Omega)\right) : \|y - \omega|_{t \in J_k}\|_{L^\infty\left(J_k; H_0^1(\Omega)\right)} \le \rho_k \right\}.$$

[5]Such a constant C_p can be estimated rigorously (see, e.g., Lemma 2 in [24]).

Finally, the mild solution of (1) is enclosed in $B(\omega)$, which is defined in (2). We call this process a "*concatenation scheme of verified numerical inclusion.*"

Remark 3.1. It is proved that the above mild solution u of (1) is in $C(J; L^2(\Omega))$, because $u(t) \in H_0^1(\Omega) \subset L^2(\Omega)$ for $t \in J$ and $f(u) \in L^1(J; L^2(\Omega))$.

3.2. Computational example. In this subsection, we present two illustrative computational examples. All of the computations are carried out on Cent OS 6.3 with a 3.10 GHz Intel Xeon E5-2687W, using MATLAB 2013a with the INTLAB toolbox for verified numerical computations, version 7.1 [26]. Let $\Omega = \{(x_1, x_2) : 0 < x_i < 1,\ i = 1, 2\} \subset \mathbb{R}^2$ be a unit square domain.

Using the usual finite element procedure and a simple first order approximation of the time derivative, we employ the full discretization to obtain $\{u_k^h\}_{k\geq 1} \subset V_h$ such that

$$\left(\frac{u_k^h - u_{k-1}^h}{\tau_k}, v_h \right)_{L^2} + a(u_k^h, v_h)_{L^2} = (f(u_k^h), v_h)_{L^2}$$

and $(\nabla u_0^h, \nabla v_h)_{L^2} = (\nabla u_0, \nabla v_h)_{L^2}$ for all $v_h \in V_h$. Let $\hat{u}_k \in V_h$ $(k = 0, 1, \ldots, n)$ be a numerical approximation of u_k^h. Then, we use the quadratic conforming finite elements (P_2-elements) on the uniform mesh triangulation. Let h be the mesh size and τ_k the step size of time discretization.

First, we consider the following Fujita-type parabolic equation:

$$\begin{cases} \partial_t u - \Delta u = u^2 & \text{in } (0, \infty) \times \Omega, \\ u(t, x) = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ u(0, x) = \gamma x_1(1 - x_1)x_2(1 - x_2) & \text{in } \Omega, \end{cases} \tag{24}$$

where $\gamma > 0$ is a parameter of the initial function. It is known [6] that for sufficiently large γ, any solution of (24) blows up in finite time. Let $h = 2^{-4}$ and $\tau_k = 2^{-8}$ $(k = 1, 2, \ldots, n)$. By varying γ over the values 1, 10, 15, and 20, we try to enclose the mild solution of (24) using our concatenation scheme. If the sufficient condition of Theorem 2.7 holds, then we can obtain ε_k and ρ_k $(k = 1, 2, \ldots, n)$ satisfying

$$\|u(t_k) - \hat{u}_k\|_{H_0^1} \leq \varepsilon_k \quad \text{and} \quad \|u - \omega\|_{L^\infty\left(J_k; H_0^1(\Omega)\right)} \leq \rho_k,$$

respectively. Here, we stopped our concatenation scheme at $t = 0.5$.

Table 1 shows the time interval J_k as well as ε_k and ρ_k for $\gamma = 1$. As seen in Table 1, the values of ε_k and ρ_k gradually increase until $t = 0.05078125$. After this time the error estimates begin to decrease. It seems that the dissipative property of this parabolic equation causes this decreasing of the error estimates. The concatenation scheme succeeds in enclosing the mild solution of (24), at least until $t = 0.5$.

For $\gamma = 10$, we obtained behavior similar to that for $\gamma = 1$. Following several steps of numerical verification, ρ_k achieves a peak in the interval $t \in (0.05859375, 0.0625]$. The concatenation scheme also succeeds in enclosing the mild solution of (24) until $t = 0.5$. Figure 1 displays the results of the concatenation scheme for $\gamma = 1$ and $\gamma = 10$, respectively.

Figure 2 illustrates the error estimate ρ_k for $\gamma = 15$ and $\gamma = 20$ on the semi-logarithmic scale. For $\gamma = 15$, the existence and local uniqueness of the mild solution can be verified until $t = 0.5$. The peak of the curve plotting the error estimate occurs in the interval $t \in (0.070312, 0.074219]$. On the other hand, for $\gamma = 20$ the

TABLE 1

The error estimates ε_k and ρ_k are shown for varying J_k, where $h = 2^{-4}$, $\tau_k = 2^{-8}$, and $\gamma = 1$.

$J_k = (t_{k-1}, t_k]$	ε_k	ρ_k
$(0, 0.00390625]$	0.054609	0.054609
$(0.00390625, 0.0078125]$	0.082536	0.086588
$(0.0078125, 0.01171875]$	0.10525	0.11138
$(0.01171875, 0.015625]$	0.12387	0.13168
$(0.015625, 0.01953125]$	0.13907	0.14826
$(0.01953125, 0.0234375]$	0.15134	0.16166
$(0.0234375, 0.02734375]$	0.16108	0.17231
$(0.02734375, 0.03125]$	0.16862	0.18057
$(0.03125, 0.03515625]$	0.17422	0.18674
$(0.03515625, 0.0390625]$	0.17815	0.19108
$(0.0390625, 0.04296875]$	0.1806	0.19382
$(0.04296875, 0.046875]$	0.18178	0.19518
$(0.046875, 0.05078125]$	0.18185	0.19534
$(0.05078125, 0.0546875]$	0.18096	0.19446
$(0.0546875, 0.05859375]$	0.17925	0.19268
$\vdots$	$\vdots$	$\vdots$
$(0.48828125, 0.4921875]$	0.00032167	0.00034722
$(0.4921875, 0.49609375]$	0.00030056	0.00032443
$(0.49609375, 0.5]$	0.00028082	0.00030313

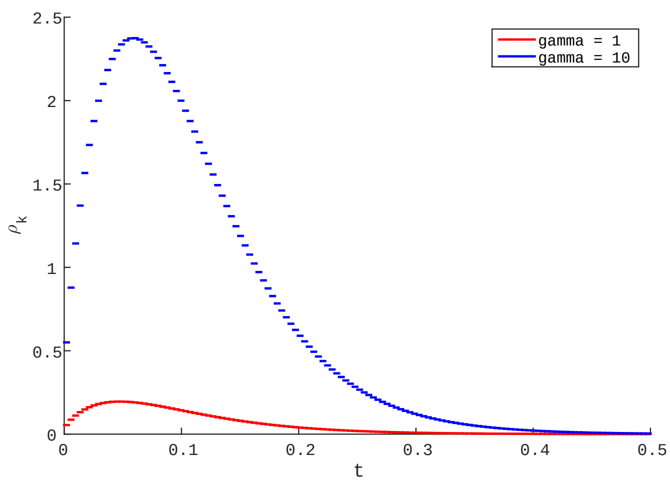

FIG. 1. *The error estimate ρ_k of (24) is plotted for $\gamma = 1$ and $\gamma = 10$. Here, $h = 2^{-4}$ and $\tau_k = 2^{-8}$.*

concatenation scheme succeeds until $t = 0.21899844010501127343459520489$. After that, the numerical verification scheme fails to enclose the mild solution.

Next, we consider another initial-boundary value problem, of the form

$$
\begin{cases} \partial_t u - \Delta u = u - u^3 & \text{in } (0,\infty)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = x_1(1-x_1)x_2(1-x_2) & \text{in } \Omega. \end{cases} \tag{25}
$$

By varying τ_k and h, we investigate the dependence of the error estimate ρ_k on the step size τ_k and the mesh size h, respectively.

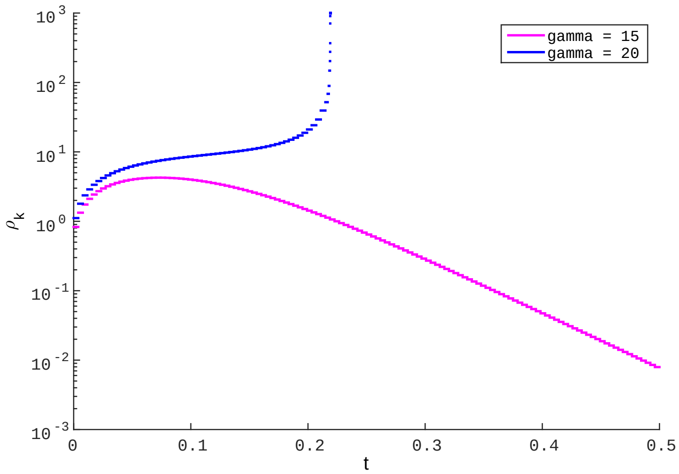

FIG. 2. *The error estimate* ρ_k *of* (24) *is plotted on the semilogarithmic scale for* $\gamma = 15$ *and* $\gamma = 20$. *Here,* $h = 2^{-4}$ *and* $\tau_k = 2^{-8}$.

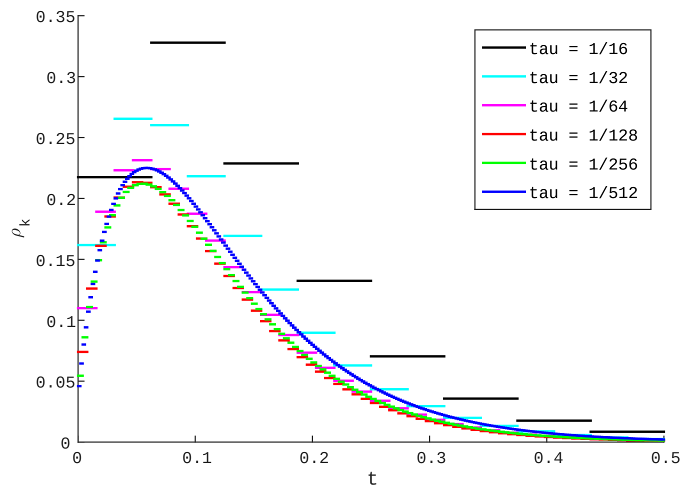

FIG. 3. *For* $\tau_k = 2^{-j}$, *where* j *takes the various values* $j = 4, 5, \ldots, 9$, *the error estimate* ρ_k *from* (25) *is plotted versus* t *when* $h = 2^{-4}$.

In Figure 3, the error estimate ρ_k $(k = 1, 2, \ldots, n)$ is plotted with $\tau_k = 2^{-j}$ $(j = 4, 5, \ldots, 9)$ and $h = 2^{-4}$.

Finally, Figure 4 illustrates the error estimate ρ_k $(k = 1, 2, \ldots, n)$ for $h = 2^{-j}$ $(j = 2, 3, 4, 5)$ and $\tau_k = 2^{-9}$.

4. Global existence proof using verified numerical computations.

4.1. Global existence theorem. In the remainder of this paper, we consider existence of a global-in-time solution. Here, a global-in-time solution is a solution of (1) that can be proved to exist in $t \in (0, \infty)$. In [13, 25], an analytic result is given that the global existence of a solution corresponds to small initial data by imposing

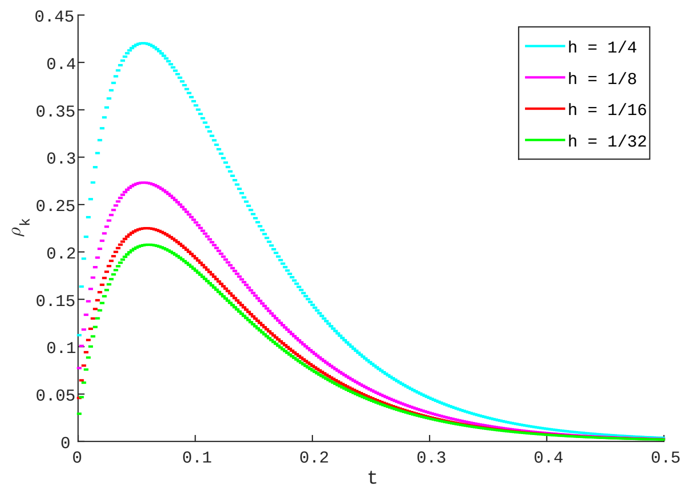

FIG. 4. *For $h = 2^{-j}$, where j takes the various values $j = 2, 3, 4, 5$, the error estimate ρ_k from* (25) *is plotted versus t when $\tau_k = 2^{-9}$.*

an assumption that $f(0) = 0$. However, with such an analytic approach it is difficult to obtain quantitative results. For example, using the analytic approach, it is difficult to answer the question of how small the initial data must be in order to obtain global-in-time solutions. In the following, we will present a method for numerically proving the existence of a global-in-time solution starting from some given initial data. To illustrate our computer-assisted method, we will provide a numerical example, proving the existence of a global-in-time solution for a certain semilinear parabolic equation, whose global existence for small initial data has already been proved.

Let $\hat{u}_n \in V_h$ for a fixed $n \in \mathbb{N}$, and suppose that our concatenation scheme succeeds in proving the existence and local uniqueness of a mild solution in $t \in (0, t_n]$. Now, we present a sufficient condition, which can be checked numerically, for the existence of a solution to the following parabolic equations:

$$\begin{cases} \partial_t u + Au = f(u) & \text{in } (t_n, \infty) \times \Omega, \\ u(t,x) = 0 & \text{on } (t_n, \infty) \times \partial\Omega, \\ u(t_n, x) = \zeta & \text{in } \Omega, \end{cases} \tag{26}$$

where ζ satisfies $\|\zeta - \hat{u}_n\|_{H_0^1} \le \varepsilon_n$.

For a fixed $\lambda > 0$, we introduce the function space

$$X_\lambda := \left\{ u \in L^\infty\left((t_n, \infty); H_0^1(\Omega)\right) : \operatorname*{ess\,sup}_{t > t_n} e^{(t-t_n)\lambda} \|u(t)\|_{H_0^1} < \infty \right\},$$

which becomes a Banach space with the norm $\|u\|_{X_\lambda} := \operatorname{ess\,sup}_{t>t_n} e^{(t-t_n)\lambda} \|u(t)\|_{H_0^1}$.

THEOREM 4.1. *For the first Fréchet derivative of $f : H_0^1(\Omega) \to L^2(\Omega)$, we assume that a nondecreasing function $\tilde{L} : \mathbb{R} \to \mathbb{R}$ exists such that*

$$\|f'[y]w\|_{L^\infty((t_n,\infty);L^2(\Omega))} \le \tilde{L}\left(\|y\|_{L^\infty\left((t_n,\infty);H_0^1(\Omega)\right)}\right) \|w\|_{H_0^1} \quad \forall w \in H_0^1(\Omega), \tag{27}$$

for any $y \in L^\infty((t_n, \infty); H_0^1(\Omega))$. Moreover, we impose the condition that $f(0) = 0$.

Then, for a fixed $0 < \lambda < \frac{\lambda_{\min}}{2}$*, if there exists* $\rho > 0$ *such that*

$$\frac{M}{\mu}\|\zeta\|_{H_0^1} + \frac{\tilde{L}(\rho)\rho}{\mu}\sqrt{\frac{2M\pi}{e(\lambda_{\min} - 2\lambda)}} < \rho, \tag{28}$$

then a solution $u(t)$ *of* (26) *uniquely exists for* $t \in (t_n, \infty)$*. Furthermore, the following estimate holds:*

$$\|u(t)\|_{H_0^1} \le \rho e^{-(t-t_n)\lambda}, \quad t \in (t_n, \infty).$$

Remark 4.2. Assume that the concatenation scheme succeeds in enclosing a mild solution $u(t)$ for $t \in (0, t_n]$, and also that the sufficient condition (28) holds for $\zeta = u(t_n)$. Then, Theorem 4.1 proves the existence of a global-in-time solution for (1).

Proof. We define an operator $S : L^\infty((t_n, \infty); H_0^1(\Omega)) \to L^\infty((t_n, \infty); H^{-1}(\Omega))$ as

$$(Su)(t) := e^{-(t-t_n)\mathcal{A}}\zeta + \int_{t_n}^{t} e^{-(t-s)\mathcal{A}} f(u(s))ds, \quad t \in (t_n, \infty).$$

Let $U := \{u \in L^\infty((t_n, \infty); H_0^1(\Omega)) : \|u\|_{X_\lambda} \le \rho\}$ for $\rho > 0$. Now, we will derive a sufficient condition for S to have a fixed point in U on the basis of Banach's fixed-point theorem. For $u \in U$, Corollary 2.5, (6), and (11) yield that

$$\begin{aligned}
\|(Su)(t)\|_{H_0^1} &\le \mu^{-1}\|\mathcal{A}(Su)(t)\|_{H^{-1}} \\
&\le \mu^{-1}\left\|e^{-(t-t_n)\mathcal{A}}\mathcal{A}\zeta\right\|_{H^{-1}} + \int_{t_n}^{t}\left\|\mathcal{A}^{\frac{1}{2}}e^{-(t-s)\mathcal{A}}\mathcal{A}^{\frac{1}{2}}f(u(s))\right\|_{H^{-1}} ds \\
&\le \mu^{-1}Me^{-(t-t_n)\lambda_{\min}}\|\zeta\|_{H_0^1} \\
&\quad + \mu^{-1}M^{\frac{1}{2}}e^{-\frac{1}{2}}\int_{t_n}^{t}(t-s)^{-\frac{1}{2}}e^{-\frac{(t-s)}{2}\lambda_{\min}}\|f(u(s))\|_{L^2} ds.
\end{aligned}$$

Because of the condition that $f(0) = 0$, the mean-value theorem implies that

$$\|f(u(s))\|_{L^2} = \|f(u(s)) - f(0)\|_{L^2} = \left\|\int_0^1 f'[\theta u(s)]u(s)d\theta\right\|_{L^2}.$$

For $y(s) = \theta u(s) \in H_0^1(\Omega)$ for each s and for each θ, it follows from (27) that

$$\|f(u(s))\|_{L^2} \le \tilde{L}\left(\|y\|_{L^\infty\left((t_n,\infty);H_0^1(\Omega)\right)}\right)\|u(s)\|_{H_0^1} \le \tilde{L}(\rho)\|u(s)\|_{H_0^1}.$$

Then, we have

$$\begin{aligned}
&\|(Su)(t)\|_{H_0^1} \\
&\le \mu^{-1}Me^{-(t-t_n)\lambda_{\min}}\|\zeta\|_{H_0^1} \\
&\quad + \mu^{-1}M^{\frac{1}{2}}e^{-\frac{1}{2}}\tilde{L}(\rho)\int_{t_n}^{t}(t-s)^{-\frac{1}{2}}e^{-\frac{(t-s)}{2}\lambda_{\min}}\|u(s)\|_{H_0^1}ds \\
&= \mu^{-1}Me^{-(t-t_n)\lambda_{\min}}\|\zeta\|_{H_0^1} \\
&\quad + \mu^{-1}M^{\frac{1}{2}}e^{-\frac{1}{2}}\tilde{L}(\rho)\int_{t_n}^{t}(t-s)^{-\frac{1}{2}}e^{-\frac{(t-s)}{2}\lambda_{\min}}e^{-(s-t_n)\lambda}\left(e^{(s-t_n)\lambda}\|u(s)\|_{H_0^1}\right) ds.
\end{aligned}$$

Thus, it holds that

$$
\begin{aligned}
(29) \qquad & e^{(t-t_n)\lambda}\|(Su)(t)\|_{H_0^1} \\
& \leq \frac{M}{\mu} e^{-(t-t_n)(\lambda_{\min}-\lambda)}\|\zeta\|_{H_0^1} + \frac{\tilde{L}(\rho)}{\mu}\sqrt{\frac{M}{e}}\|u\|_{X_\lambda}\int_{t_n}^{t}(t-s)^{-\frac{1}{2}}e^{-(t-s)\frac{\lambda_{\min}-2\lambda}{2}}ds.
\end{aligned}
$$

Now, let Γ be the Gamma function. From $u \in U$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and (28) we have

$$
\|S(u)\|_{X_\lambda} \leq \frac{M}{\mu}\|\zeta\|_{H_0^1} + \frac{\tilde{L}(\rho)\rho}{\mu}\sqrt{\frac{2M\pi}{(\lambda_{\min}-2\lambda)e}} < \rho. \tag{30}
$$

This implies that $S(u) \in U$.

Next, let $\varphi, \psi \in U$. It follows from the mean-value theorem that

$$
\|f(\varphi(s)) - f(\psi(s))\|_{L^2} = \left\|\int_0^1 f'[\theta\varphi(s) + (1-\theta)\psi(s)](\varphi(s)-\psi(s))d\theta\right\|_{L^2}.
$$

Then, for $y(s) = \theta\varphi(s) + (1-\theta)\psi(s) \in H_0^1(\Omega)$ for each s and for each θ, we have

$$
\|f(\varphi(s)) - f(\psi(s))\|_{L^2} \leq \tilde{L}(\rho)\|\varphi(s) - \psi(s)\|_{L^2}.
$$

From this, it follows that

$$
\begin{aligned}
& \|S(\varphi(t)) - S(\psi(t))\|_{H_0^1} \\
& \leq \mu^{-1}M^{\frac{1}{2}}e^{-\frac{1}{2}}\int_{t_n}^{t}(t-s)^{-\frac{1}{2}}e^{-\frac{(t-s)}{2}\lambda_{\min}}\|f(\varphi(s)) - f(\psi(s))\|_{L^2}\,ds \\
& \leq \mu^{-1}M^{\frac{1}{2}}e^{-\frac{1}{2}}\tilde{L}(\rho)\int_{t_n}^{t}(t-s)^{-\frac{1}{2}}e^{-\frac{(t-s)}{2}\lambda_{\min}}\|\varphi(s) - \psi(s)\|_{H_0^1}\,ds \\
& = \mu^{-1}M^{\frac{1}{2}}e^{-\frac{1}{2}}\tilde{L}(\rho)\int_{t_n}^{t}(t-s)^{-\frac{1}{2}}e^{-\frac{(t-s)}{2}\lambda_{\min}}e^{-(s-t_n)\lambda}(e^{(s-t_n)\lambda}\|\varphi(s) - \psi(s)\|_{H_0^1})ds \\
& \leq \frac{\tilde{L}(\rho)}{\mu}\sqrt{\frac{M}{e}}\|\varphi-\psi\|_{X_\lambda}\int_{t_n}^{t}(t-s)^{-\frac{1}{2}}e^{-\frac{(t-s)}{2}\lambda_{\min}}e^{-(s-t_n)\lambda}ds.
\end{aligned}
$$

By applying procedures similar to those used to derive the estimates in (29) and (30), we obtain that

$$
\|S(\varphi) - S(\psi)\|_{X_\lambda} \leq \frac{\tilde{L}(\rho)}{\mu}\sqrt{\frac{2M\pi}{(\lambda_{\min}-2\lambda)e}}\|\varphi - \psi\|_{X_\lambda}.
$$

From the assumption in (28), it follows that $\frac{\tilde{L}(\rho)}{\mu}\sqrt{\frac{2M\pi}{(\lambda_{\min}-2\lambda)e}} < 1$ holds. Therefore, S is a contraction mapping on U. Then, Banach's fixed-point theorem states that there exists a fixed point $u \in U$. Furthermore, it follows from the definition of X_λ that the following holds for $t \in (t_n, \infty)$:

$$
\|u(t)\|_{H_0^1} \leq \rho e^{-(t-t_n)\lambda}. \qquad \square
$$

Table 2
The existence of a global-in-time solution for (24) *is proved after the nth time verification when* $\lambda = 0.5$, $h = 2^{-4}$, *and* $\tau_k = 2^{-8}$.

γ	n	t_n	ρ
until 18	1	0.00390625	(<) 6.4291
19	11	0.160156	4.7269
20	12	0.175781	4.3507
21	13	0.191406	4.1088
22	14	0.207031	4.0671
23	15	0.222656	4.4724
24	17	0.253906	4.4471

4.2. Computational results. Let us again consider the Fujita-type parabolic problem (24) from section 3.2. Here, we fix $\lambda = 0.5$, $h = 2^{-4}$, and $\tau_k = 2^{-8}$ ($k = 1, 2, \ldots$). When the numerical verification based on Theorem 2.7 succeeds in enclosing a mild solution in $t \in (t_{k-1}, t_k]$, we check whether the sufficient condition (28) holds. If it holds, then we set $t_n = t_k$. In this case, the existence of a global-in-time solution for (24) is proved, because the existence and local uniqueness of the mild solution has already been proved in $t \in (0, t_n]$. Otherwise, we continue the numerical verification to the next step, i.e., $t \in (t_k, t_{k+1}]$.

Table 2 presents the results of applying our method from section 4.1 to (24). For $\gamma = 1, 2, \ldots, 18$, the existence of the global-in-time solution is proved by checking the sufficient condition of Theorem 4.1 at $t_1 = 0.00390625$. For $\gamma = 19, 20, \ldots, 24$, the existence of mild solutions for (24) is verified at some at t_n with $n > 1$. For $\gamma > 24$, the concatenation scheme fails because the error estimate becomes too large to satisfy the sufficient condition of Theorem 2.7.

5. Conclusion. We conclude this paper by summarizing our results and discussing some potential extensions. We have proposed a novel method based on verified numerical computations to verify the existence and local uniqueness of mild solutions for initial-boundary value problems of semilinear parabolic partial differential equations, using semigroup theory. Theorem 2.7 presents a sufficient condition for a mild solution to be enclosed in a ball centered at a numerical solution ω with a radius $\rho > 0$. We have also presented a *concatenation scheme of verified numerical inclusion*, to extend the time interval in which the existence of a mild solution is guaranteed. If the concatenation scheme succeeds in proving the existence and local uniqueness of a mild solution for $t \in (0, t_n]$, then Theorem 4.1 provides a sufficient condition for the existence of a global-in-time solution for a certain semilinear parabolic equation.

The proposed method could be generalized to semilinear parabolic equations for a bounded polyhedral domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) of the form

$$\begin{cases} \partial_t u + Au = f(u) & \text{in } (0, \infty) \times \Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega \end{cases}$$

for a triplet of Hilbert spaces such that $V \subset X \subset V^*$, with appropriate boundary conditions corresponding to A, where $A : D(A) \subset V \to H$ ($= X$ or V^*) is a self-adjoint differential operator on the space variable generating an analytic semigroup $\{e^{-tA}\}_{t \geq 0}$. Moreover, ∂_t denotes $\frac{\partial}{\partial t}$, $f : V \to X$ is a certain nonlinear map, and $u_0 \in X$ is a given initial function.

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