NUMERICAL VERIFICATION FOR EXISTENCE OF A GLOBAL-IN-TIME SOLUTION TO SEMILINEAR PARABOLIC EQUATIONS

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ABSTRACT. This paper presents a method of numerical verification for the existence of a global-in-time solution to a class of semilinear parabolic equations. Such a method is based on two main theorems in this paper. One theorem gives a sufficient condition for proving the existence of a solution to the semilinear parabolic equations with the initial point $t = t' \geq 0$. If the sufficient condition does not hold, the other theorem is used for enclosing the solution for time $t \in (0, \tau]$, $\tau > 0$ in a neighborhood of a numerical solution. Numerical results of obtaining a global-in-time solution for a certain semilinear parabolic equation are also given.

CONTENTS

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1. Introduction

Let Ω be a bounded and convex domain in $\mathbb{R}^2$. We consider the existence of a global-in-time solution [1] for the following semilinear parabolic equations:

$$\begin{cases} \partial_t u - \Delta u = f(x,u), & t \in (0,\infty),\ x \in \Omega, \quad \text{(1a)} \\ u = 0, & t \in (0,\infty),\ x \in \partial\Omega, \quad \text{(1b)} \\ u(0,x) = u_0(x), & x \in \Omega, \quad \text{(1c)} \end{cases}$$

where $\partial_t u = \frac{\partial u}{\partial t}$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian, whose domain is $\mathcal{D}(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, $u_0 \in H_0^1(\Omega)$ is an initial function, and f is a function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$, and it maps from $H_0^1(\Omega)$ into $L^2(\Omega)$ in the sense that $f(\cdot, v) \in L^2(\Omega)$ for each $v \in H_0^1(\Omega)$. The operator $f : H_0^1(\Omega) \to L^2(\Omega)$ defined in this sense is assumed to be twice Fréchet differentiable. Unless otherwise specified, $f'[v]$ and $f''[v]$ denote the first and the second order Fréchet derivatives of f at $v \in H_0^1(\Omega)$ as assuming that f is an operator from $H_0^1(\Omega)$ to $L^2(\Omega)$, respectively. The main aim of this paper is to present Theorem 3.1 in subsection 3.1 and Theorem 3.2 in subsection 3.2. Then, we propose an algorithm for numerically verifying the existence of a global-in-time solution to (1).

There have been many studies on the existence of global-in-time solutions for some parabolic equations related to (1). As a pioneer work, for the parabolic equation (1a) and (1c) when $f(x,u) = u^p$ ($p \in \mathbb{R}$) and $\Omega = \mathbb{R}^m$ ($m \in \mathbb{N}$), H. Fujita has found an exponent concerning the existence of a global-in-time solution in 1966 [1]. Then, studies of solutions to various parabolic equations have been developed in the field of mathematical analysis ([2, 3, 4, 5], etc). In particular, for the parabolic equation (1), there exist analytical studies concerning the global-in-time solution that converges to the zero function ([6, 7, 8], etc). In this paper, we cite the following theorem:

Theorem 1.1 (c.f. Theorem 19.2 in [9])**.** *Let us consider*

$$\begin{cases} \partial_t u - \Delta u = f(u), & t \in (0,\infty),\ x \in \Omega, \quad \text{(2a)} \\ u = 0, & t \in (0,\infty),\ x \in \partial\Omega, \quad \text{(2b)} \\ u(0,x) = u_0(x), & x \in \Omega, \quad \text{(2c)} \end{cases}$$

where the domain Ω is bounded, $u_0 \in L^\infty(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function such that $f(0) = 0$ and $f'(0) < \lambda_{\min}$. Here, $\lambda_{\min}$ denotes the smallest eigenvalue of $-\Delta$. There are constants $\nu > 0$, $\eta > 0$, and $K \geq 1$ such that, for all $u_0 \in L^\infty(\Omega)$ with $\|u_0\|_{L^\infty} \leq \eta$, there exists a solution u of (2) satisfying

$$\|u(t,\cdot)\|_{L^\infty} \leq \tilde{\rho} e^{-\nu t},\ t \in (0,\infty), \tag{3}$$

where $\tilde{\rho} = K\|u_0\|_{L^\infty}$.

The main aim of this paper is to give a method of calculating the values ν (> 0) and $\tilde{\rho}$ (> 0) appearing to (3). In order to calculate these values, this paper presents a verification algorithm. The algorithm tries to enclose a solution that exponentially converges to a stationary solution of (1) by numerically checking whether sufficient conditions in Theorem 3.1 and Theorem 3.2 hold, respectively.

[1] A solution that exists for $t \in (0,\infty)$ is called a global-in-time solution. We consider the solution of (1) in $L^\infty((0,\infty); H_0^1(\Omega))$ in this paper.

M.T. Nakao, T. Kinoshita, and T. Kimura have proposed a computer-assisted method for enclosing a solution to a class of parabolic equations based on verified numerical computations [10, 11, 12]. Their method is based on estimating a norm of an inverse operator related to the parabolic equations. Moreover, S. Cai [13] has derived a sufficient condition that is related to the existence of a global-in-time solution for time $t > t'$, $t' \geq 0$ to a system of reaction-diffusion equations through verified numerical computations using an analytic semigroup over $L^\infty(\Omega) \times L^\infty(\Omega)$.

Recently, we have developed a method for verifying the existence of a solution to a semilinear parabolic equation by using an analytic semigroup over $H^{-1}(\Omega)$ (a topological dual space of $H^1_0(\Omega)$) in [14]. In this paper, by using an analytic semigroup over $L^2(\Omega)$, we provide a verification algorithm for enclosing a mild solution of (1), whose definition is given in Section 2. This algorithm is expected to enclose the solution of (1) more tightly than results in the previous paper. This is because a residual estimate obtained by the semigroup over $L^2(\Omega)$ in this paper is also expected to be tighter than one obtained by the semigroup over $H^{-1}(\Omega)$ in the previous paper. The comparison of the residual estimates is given in Appendix A. We will show a method for verifying the existence of a global-in-time solution. In such a method, the existence of a global-in-time solution for (1) is shown by the following procedure: First, we check whether the sufficient condition in Theorem 3.1 holds. If this condition holds, we can show the existence of a global-in-time solution. Otherwise, we try to enclose a mild solution $u(t)$ for $t \in (0, \tau]$, $\tau > 0$ in a neighborhood of a numerical solution to (1) by checking whether (16) in Theorem 3.2 holds. If the enclosure of the solution is obtained, we also verify the existence of the mild solution $u(t)$ for $t \in (\tau, \infty)$ by using Corollary 3.3 and Theorem 3.1. By Algorithm 1 based on Theorem 3.1, Theorem 3.2, and Corollary 3.3, the existence of a global-in-time solution for (1) is expected to be guaranteed in a subset of the Banach space $L^\infty\left((0,\infty); H^1_0(\Omega)\right)$.

The organization of this paper is given as follows: In Section 2, we give preliminaries throughout this paper. In subsection 3.1, Theorem 3.1 gives a sufficient condition for verifying the existence of a solution to (1) with the initial point $t = 0$ replaced by some $t = t' > 0$. In subsection 3.2, a verification algorithm is given for showing the existence of a global-in-time solution. The procedure of the verification algorithm is described in Algorithm 1. In Section 4, we give numerical results of verifying the existence of a global-in-time solution to certain semilinear parabolic equations. We present some quantification of an analytical result using the verification algorithm. In appendixes, we give several estimates, which will be useful in order to check the existence of a global-in-time solution to (1).

2. Preliminaries

The inner product of $L^2(\Omega)$ is given by

$$(u, v)_{L^2} := \int_\Omega u(x)v(x)dx.$$

The norm of $L^2(\Omega)$ is defined by $\|u\|_{L^2(\Omega)} := \sqrt{(u,u)_{L^2}}$. For a positive integer m, let $H^m(\Omega)$ be the mth order Sobolev space of $L^2(\Omega)$. We define a function space $H^1_0(\Omega) := \{u \in H^1(\Omega) | u = 0 \text{ on } \partial\Omega\}$, where $u = 0$ on $\partial\Omega$ is in the trace sense. We use the norm of $H^1_0(\Omega)$ such that $\|u\|_{H^1_0} := \|\nabla u\|_{L^2}$.

Let J be an interval in $(0,\infty)$. Let a function space Y be a Banach space with the norm $\|\cdot\|_Y$. We define a function space $L^\infty(J;Y)$ as

$$L^\infty(J;Y) := \left\{ u : J\times\Omega \to \mathbb{R} \mid u(t,\cdot)\in Y,\ \operatorname*{ess\,sup}_{t\in J} \|u(t,\cdot)\|_Y < \infty \right\}$$

with the norm $\|u\|_{L^\infty(J;Y)} := \operatorname{ess\,sup}_{t\in J}\|u(t,\cdot)\|_Y$. Let $C^0(J)$ be the function space of all continuous functions from J to $\mathbb{R}$. We also define a function space $C^0(J;Y)$ as

$$C^0(J;Y) := \left\{ u : J\times\Omega \to \mathbb{R} \mid u(t,\cdot)\in Y,\ \|u(t,\cdot)\|_Y \in C^0(J) \right\}.$$

Let P and Q be Banach spaces. For a bounded operator $B : P\to Q$, the operator norm of B is denoted by $\|B\|_{P,Q}$.

We denote $A = -\Delta : \mathcal{D}(A)\to L^2(\Omega)$ and $\mathcal{D}(A) = H^2(\Omega)\cap H^1_0(\Omega)$. We define $\rho(A)$ as a resolvent set of A:

$$\rho(A) := \{z\in\mathbb{C} \mid (zI-A)^{-1} : L^2(\Omega)\to L^2(\Omega) \text{ exists and is a bounded operator.}\}.$$

Let $\sigma(A) = \mathbb{C}\setminus\rho(A)$ and $\lambda_{\min}$ denotes the minimum value of $\sigma(A)$. For $0\le\alpha\le 1$, a fractional operator of A is defined by

$$A^\alpha u := \sum_{j=1}^\infty \lambda_j^\alpha c_j\psi_j,\ \mathcal{D}(A^\alpha) = \left\{ u = \sum_{j=1}^\infty c_j\psi_j \in L^2(\Omega) \mid \sum_{j=1}^\infty c_j^2\lambda_j^{2\alpha} < \infty \right\},$$

where $\{\psi_j\}_{j\in\mathbb{N}}$ is the complete orthonormal basis of eigenfunctions of A in $L^2(\Omega)$, $c_j = (u,\psi_j)_{L^2}$, and $\{\lambda_j\}_{j\in\mathbb{N}} = \sigma(A)$.

It is known that $-A$ generates the analytic semigroup $\left\{e^{-tA}\right\}_{t\ge 0}$ over $L^2(\Omega)$ (see e.g., [15, 16]).

Definition 2.1. Let $J = (t_0,t_1]$ $(0\le t_0 < t_1 \le \infty)$. For the semilinear parabolic equation:

$$\begin{cases} \partial_t u - \Delta u = f(x,u), & t\in J, x\in\Omega, \\ u(t,x) = 0, & t\in J, x\in\partial\Omega, \\ u(t_0,x) = u_0(x), & x\in\Omega, \end{cases} \tag{4}$$

the function $u\in C^0(J;L^2(\Omega))$ given by

$$u(t) = e^{-(t-t_0)A}u_0 + \int_{t_0}^t e^{-(t-s)A} f(\cdot,u(s))ds \ (t\in J)$$

is a mild solution of (4) on J.

We introduce Lemma 2.2 and Lemma 2.3 (see e.g., [15, 16]).

Proposition 2.2. *$\mathcal{D}(A^{1/2}) = H^1_0(\Omega)$ and*

$$\|w\|_{H^1_0} = \|A^{1/2}w\|_{L^2},\ \forall w\in H^1_0(\Omega) \tag{5}$$

hold.

Proposition 2.3. *Let $\alpha\in(0,1]$. If $u\in\mathcal{D}(A^\alpha)$, then,*

$$A^\alpha e^{-tA}u = e^{-tA}A^\alpha u,\ t>0$$

holds.

Furthermore, we obtain the following lemma:

Proposition 2.4. *Let $\lambda_{\min}$ be the minimum eigenvalue of A. For fixed $\alpha \in (0,1)$ and $\beta \in (0,1)$, the following estimate holds:*

$$\|A^\alpha e^{-tA}\|_{L^2,L^2} \le \left(\frac{\alpha}{e\beta}\right)^\alpha t^{-\alpha} e^{-(1-\beta)t\lambda_{\min}},\ t>0. \tag{6}$$

Proof. Since the minimum eigenvalue of A is positive, we have

$$\sup_{x\in(\lambda_{\min},\infty)} x^\alpha e^{-\beta t x} \le \left(\frac{\alpha}{e\beta t}\right)^\alpha \quad \text{and} \quad \sup_{x\in(\lambda_{\min},\infty)} e^{-(1-\beta)tx} \le e^{-(1-\beta)t\lambda_{\min}}$$

for fixed $\alpha \in (0,1)$ and $\beta \in (0,1)$. From the spectral mapping theorem the following inequality holds:

$$\begin{aligned}\|A^\alpha e^{-tA}\|_{L^2,L^2} &\le \sup_{x\in(\lambda_{\min},\infty)} x^\alpha e^{-tx}\\ &\le \sup_{x\in(\lambda_{\min},\infty)} x^\alpha e^{-\beta t x} \sup_{x\in(\lambda_{\min},\infty)} e^{-(1-\beta)tx}.\end{aligned}$$

This indicates that the inequality (6) holds. □

For $x>0$, the error function $\operatorname{erf}(x)$ is defined by

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds.$$

By an elemental calculation it follows for $\alpha>0$ and $x>0$,

$$\int_0^x s^{-1/2} e^{-\alpha s} ds = \sqrt{\frac{\pi}{\alpha}}\operatorname{erf}(\sqrt{\alpha x}). \tag{7}$$

Let $\rho>0$ and J be any interval in $(0,\infty)$. For $v \in L^\infty(J;H_0^1(\Omega))$, a closed ball $B_{L^\infty(J;H_0^1(\Omega))}(v,\rho)$ is defined by

$$B_{L^\infty(J;H_0^1(\Omega))}(v,\rho) := \left\{ y \in L^\infty\left(J;H_0^1(\Omega)\right) \middle| \|y-v\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \le \rho \right\}.$$

3. Numerical verification for a global-in-time solution

3.1. Global-in-time existence theorem. Let $\phi \in \mathcal{D}(A)$ be a stationary solution of (1). Namely, ϕ satisfies

$$\begin{cases} A\phi(x) = f(x,\phi(x)), & x\in\Omega,\\ \phi(x) = 0, & x \in \partial\Omega.\end{cases}$$

A function space V_h denotes a finite dimensional subspace[2] of $\mathcal{D}(A)$ depending on a parameter $h>0$. We assume that ϕ is a locally unique stationary solution in the ball [3] :

$$B_{H_0^1}(\hat\phi,\rho') := \left\{\mu \in H_0^1(\Omega) \middle| \|\mu-\hat\phi\|_{H_0^1} \le \rho'\right\} \text{ for } \rho'>0, \tag{8}$$

where $\hat\phi \in V_h$ is a certain numerical approximation of ϕ.

In this subsection, we give an inequality that provides a sufficient condition of enclosing a mild solution $u(t)$ of (1) with the initial point $t=0$ replaced by some

[2]For example, V_h is a C^1-finite element subspace. Alternatively, when Ω is a rectangular domain, V_h is spanned by the Fourier bases.

[3]One can easily check whether a stationary solution ϕ uniquely exists in $B_{H_0^1}(\hat\phi,\rho')$ by using various computer-assisted methods, e.g., [17, 18, 19].

$t = t' > 0$ in a neighborhood of the stationary solution ϕ. In the following, we consider a mild solution of

$$\begin{cases} \partial_t u + Au = f(x,u), & t \in (t',\infty),\ x \in \Omega, \\ u = 0, & t \in (t',\infty),\ x \in \partial\Omega, \\ u(t',x) = \eta, & x \in \Omega, \end{cases} \tag{9}$$

satisfying

$$u(t) = e^{-(t-t')A}\eta + \int_{t'}^{t} e^{-(t-s)A} f(\cdot, u(s))ds,$$

where $\eta \in B_{H_0^1}(\hat{u}, \varepsilon)$ for a certain $\hat{u} \in V_h$.

For a fixed $\lambda \geq 0$ we define a function space X_λ as

$$X_\lambda := \left\{ u \in L^\infty((t',\infty); H_0^1(\Omega)) \middle| \operatorname*{ess\,sup}_{t\in(t',\infty)} e^{(t-t')\lambda} \|u(t,\cdot)\|_{H_0^1} < \infty \right\},$$

where X_λ becomes a Banach space with the norm

$$\|u\|_{X_\lambda} := \operatorname*{ess\,sup}_{t\in(t',\infty)} e^{(t-t')\lambda} \|u(t,\cdot)\|_{H_0^1}.$$

The following theorem gives a sufficient condition for enclosing the mild solution of (9) in X_λ. This theorem gives quantification of the analytical result corresponding to the estimate (3). Some examples are also given in Section 4.

Theorem 3.1. *We consider the semilinear parabolic equation* (9)*. We assume that $\phi \in \mathcal{D}(A)$ is a locally unique stationary solution of* (9) *in $B_{H_0^1}(\hat{\phi}, \rho')$. We also assume that there exists a non-decreasing function $L_\phi : \mathbb{R} \to \mathbb{R}$ such that for $y \in B_{L^\infty((t',\infty);H_0^1(\Omega))}(\phi, \rho)$*

$$\|f'[y]u\|_{L^\infty((t',\infty);L^2(\Omega))} \leq L_\phi(\rho)\|u\|_{L^\infty((t',\infty);H_0^1(\Omega))},\ \forall u \in L^\infty((t',\infty); H_0^1(\Omega)), \tag{10}$$

where the function L_ϕ depends on ϕ. Let λ satisfy $0 \leq \lambda < \lambda_{\min}/2$. If there exists $\rho > 0$ such that

$$\|\eta - \phi\|_{H_0^1} + L_\phi(\rho)\rho\sqrt{\frac{2\pi}{e(\lambda_{\min} - 2\lambda)}} < \rho, \tag{11}$$

then, a mild solution $u(t)$ of (9) *uniquely exists in*

$$B_{L^\infty((t',\infty);H_0^1(\Omega))}(\phi, \rho) := \left\{ u \in X_\lambda \,|\, \|u - \phi\|_{X_\lambda} \leq \rho \right\}.$$

Therefore, the following estimate holds:

$$\|u(t) - \phi\|_{H_0^1} \leq \rho e^{-(t-t')\lambda},\ t \in (t',\infty).$$

Remark. The non-decreasing function L_ϕ is essential for our verification method because the existence of $\rho > 0$ satisfying (11) highly depends on the L_ϕ. For example there exists L_ϕ given in (10) if f is a polynomial, i.e. $f(x,u) = \sum_{i=1}^{N} c_i u^i$, where $N \in \mathbb{N}$ and $c_i \in \mathbb{R}$. However such a non-decreasing function L_ϕ does not exist if $f(x,u) = u^{1/2}$.

Remark. Since $\|\eta-\hat{u}\|_{H_0^1}\le\varepsilon$ and a stationary solution ϕ exists in $B_{H_0^1}(\hat{\phi},\rho')$, it follows

$$\begin{aligned}\|\eta-\phi\|_{H_0^1} &\le \|\eta-\hat{u}\|_{H_0^1}+\|\hat{u}-\hat{\phi}\|_{H_0^1}+\|\hat{\phi}-\phi\|_{H_0^1}\\ &\le \varepsilon+\|\hat{u}-\hat{\phi}\|_{H_0^1}+\rho',\end{aligned}$$

where we remark that $\|\hat{u}-\hat{\phi}\|_{H_0^1}$ is rigorously computable by using interval arithmetic. Therefore, $\|\eta-\phi\|_{H_0^1}$ in Theorem 3.1 can be estimated rigorously.
Proof of Theorem 3.1.
A nonlinear operator $S : L^\infty\left((t',\infty);H_0^1(\Omega)\right)\to L^\infty\left((t',\infty);H_0^1(\Omega)\right)$ is defined by

$$(Sz)(t):=e^{-(t-t')A}(\eta-\phi)+\int_{t'}^{t}e^{-(t-s)A}\left(f(\cdot,z(s)+\phi)-f(\cdot,\phi)\right)ds,\ t\in(t',\infty).$$

We note that the solution $u(t):=z(t)+\phi$ is a mild solution of (9) if and only if z is a fixed point of S. Let $Z:=\{z\in X_\lambda | \|z\|_{X_\lambda}\le\rho\}$ for a certain $\rho>0$. We derive a condition based on Banach's fixed-point theorem so that S has a fixed-point in Z.

Let $z\in Z$. Then, (5) yields

$$\begin{aligned}&e^{(t-t')\lambda}\|(Sz)(t)\|_{H_0^1}\\ &\le e^{(t-t')\lambda}\|e^{-(t-t')A}(\eta-\phi)\|_{H_0^1}\\ &\ + e^{(t-t')\lambda}\int_{t'}^{t}\|e^{-(t-s)A}\left(f(\cdot,z(s)+\phi)-f(\cdot,\phi)\right)\|_{H_0^1}ds\\ &= e^{(t-t')\lambda}\|A^{\frac{1}{2}}e^{-(t-t')A}(\eta-\phi)\|_{L^2}\\ &\ + e^{(t-t')\lambda}\int_{t'}^{t}\|A^{\frac{1}{2}}e^{-(t-s)A}\left(f(\cdot,z(s)+\phi)-f(\cdot,\phi)\right)\|_{L^2}ds\\ &\le e^{(t-t')\lambda}\|A^{1/2}e^{-(t-t')A}(\eta-\phi)\|_{L^2}\\ &\ + \int_{t'}^{t}e^{(t-s)\lambda}\|A^{1/2}e^{-(t-s)A}\|_{L^2,L^2}e^{(s-t')\lambda}\|\left(f(\cdot,z(s)+\phi)-f(\cdot,\phi)\right)\|_{L^2}ds.\end{aligned}$$

Since $\lambda < \lambda_{\min}/2$ and $\int_{t'}^{t}(t-s)^{-1/2}e^{-(t-s)(\lambda_{\min}-2\lambda)/2}ds < \infty$ for a fixed $t > t'$ hold, (5), Lemma 2.3, Lemma 2.4 with $\alpha = \beta = 1/2$, and (7) imply

$$
\begin{aligned}
& e^{(t-t')\lambda}\|(Sz)(t)\|_{H_0^1} \\
&\le e^{(t-t')\lambda}\|A^{1/2}e^{-(t-t')A}(\eta-\phi)\|_{L^2} \\
&\quad + \operatorname*{ess\,sup}_{s\in(t',\infty)}\left(e^{(s-t')\lambda}\,\|f(\cdot,z(s)+\phi)-f(\cdot,\phi)\|_{L^2}\right) \\
&\quad \times e^{-\frac{1}{2}}\int_{t'}^{t}(t-s)^{-\frac{1}{2}}e^{-(t-s)\frac{\lambda_{\min}-2\lambda}{2}}ds \\
&= e^{(t-t')\lambda}\|e^{-(t-t')A}A^{\frac{1}{2}}(\eta-\phi)\|_{L^2} \\
&\quad + \frac{\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{(\lambda_{\min}-2\lambda)(t-t')}{2}}\right)}{\sqrt{e(\lambda_{\min}-2\lambda)}}\operatorname*{ess\,sup}_{s\in(t',\infty)}\left(e^{(s-t')\lambda}\,\|f(\cdot,z(s)+\phi)-f(\cdot,\phi)\|_{L^2}\right) \\
&\le e^{(t-t')(\lambda-\lambda_{\min})}\|\eta-\phi\|_{H_0^1} \\
&\quad + \frac{\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{(\lambda_{\min}-2\lambda)(t-t')}{2}}\right)}{\sqrt{e(\lambda_{\min}-2\lambda)}}\operatorname*{ess\,sup}_{s\in(t',\infty)}\left(e^{(s-t')\lambda}\,\|f(\cdot,z(s)+\phi)-f(\cdot,\phi)\|_{L^2}\right).
\end{aligned}
$$

Fix $s \in (t',\infty)$. For $v \in L^2(\Omega)$ and $0 \le \theta \le 1$, it follows from the mean-value theorem that

$$
(f(\cdot,\phi+z(s))-f(\cdot,\phi),v)_{L^2} = \int_0^1 (f'[\phi+\theta z(s)]z(s),v)_{L^2}d\theta.
$$

For the fixed $s \in (t',\infty)$ and $v \in L^2(\Omega)$, the Schwarz inequality and (10) give

$$
\begin{aligned}
&\left|e^{(s-t')\lambda}(f(\cdot,\phi+z(s))-f(\cdot,\phi),v)_{L^2}\right| \\
&\le \int_0^1\left|(f'[\phi+\theta z(s)](e^{(s-t')\lambda}z(s)),v)_{L^2}\right|d\theta \\
&\le \int_0^1\|f'[\phi+\theta z(s)](e^{(s-t')\lambda}z(s))\|_{L^2}d\theta\|v\|_{L^2} \\
&\le L_\phi(\rho)\|z\|_{X_\lambda}\|v\|_{L^2}.
\end{aligned}
$$

Therefore, we obtain

$$
\operatorname*{ess\,sup}_{s\in(t',\infty)}\left(e^{(s-t')\lambda}\,\|f(\cdot,z(s)+\phi)-f(\cdot,\phi)\|_{L^2}\right) \le L_\phi(\rho)\|z\|_{X_\lambda}
$$

which implies

$$
e^{(t-t')\lambda}\|(Sz)(t)\|_{H_0^1} \le \|\eta-\phi\|_{H_0^1} + L_\phi(\rho)\rho\frac{\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{(\lambda_{\min}-2\lambda)(t-t')}{2}}\right)}{\sqrt{e(\lambda_{\min}-2\lambda)}}.
$$

Since $\mathrm{erf}(x)$ is a monotonically increasing function for $x > 0$ and $\mathrm{erf}(x) \to 1$ as $x \to \infty$, we have

$$
\|S(z)\|_{X_\lambda} \le \|\eta-\phi\|_{H_0^1} + L_\phi(\rho)\rho\sqrt{\frac{2\pi}{e(\lambda_{\min}-2\lambda)}}.
$$

Therefore, if $\rho > 0$ satisfies (11), $S(z) \in Z$ holds.

For any $z_1, z_2 \in Z$, it follows from $\lambda < \lambda_{\min}/2$ that

$$
\begin{aligned}
& e^{(t-t')\lambda} \left\| (Sz_1)(t) - (Sz_2)(t) \right\|_{H_0^1} \\
& \leq \int_{t'}^{t} e^{(t-s)\lambda} \| A^{1/2} e^{-(t-s)A} \|_{L^2,L^2} e^{(s-t')\lambda} \| \left(f(\cdot, z_1(s)+\phi) - f(\cdot, z_2(s)+\phi) \right) \|_{L^2} ds \\
& \leq \operatorname*{ess\,sup}_{s\in(t',\infty)} \left(e^{(s-t')\lambda} \| f(\cdot, z_1(s)+\phi) - f(\cdot, z_2(s)+\phi) \|_{L^2} \right) \\
& \quad \times e^{-1/2} \int_{t'}^{t} (t-s)^{-1/2} e^{-(t-s)\frac{\lambda_{\min}-2\lambda}{2}} ds \\
& \leq L_\phi(\rho) \| z_1 - z_2 \|_{X_\lambda} e^{-1/2} \int_{t'}^{t} (t-s)^{-1/2} e^{-(t-s)\frac{\lambda_{\min}-2\lambda}{2}} ds
\end{aligned}
$$

From (7), we obtain

$$
e^{(t-t')\lambda} \left\| (Sz_1)(t) - (Sz_2)(t) \right\|_{H_0^1} \leq L_\phi(\rho) \frac{\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{(\lambda_{\min}-2\lambda)(t-t')}{2}} \right)}{\sqrt{e(\lambda_{\min}-2\lambda)}} \| z_1 - z_2 \|_{X_\lambda}.
$$

Then, it turns out that

$$
\| S(z_1) - S(z_2) \|_{X_\lambda} \leq L_\phi(\rho) \sqrt{\frac{2\pi}{e(\lambda_{\min}-2\lambda)}} \| z_1 - z_2 \|_{X_\lambda}.
$$

If $\rho > 0$ satisfies (11), $L_\phi(\rho)\sqrt{\frac{2\pi}{e(\lambda_{\min}-2\lambda)}} < 1$ holds. Then, S becomes a contraction mapping on Z. Banach's fixed-point theorem proves that a fixed point of S uniquely exists in Z. $\square$

In order to verify the existence of a global-in-time solution to (1) we set $t' = 0$ in (9). Then, we check whether the sufficient condition in Theorem 3.1 holds. If this condition holds, we can show existence of the global-in-time solution in $L^\infty((0,\infty); H_0^1(\Omega))$. Otherwise, we try to enclose a mild solution of (1) for $t \in (0,\tau],\ 0 < \tau < \infty$ in a neighborhood of a numerical solution. Such a procedure is introduced in the next subsection.

3.2. **Verification algorithm.** For fixed t_0 and t_1 satisfying $0 \leq t_0 < t_1 < \infty$, let $J := (t_0, t_1]$ and $\tau := t_1 - t_0$. In this subsection, we give a sufficient condition for guaranteeing the existence and the local-in-time uniqueness (Theorem 3.2) of a mild solution to (1) for time $t \in J$. We also give an a posteriori error estimate in Corollary 3.3. Let $\hat{u}_0 \in V_h$ and $\hat{u}_1 \in V_h$. Then, we consider a mild solution of

$$
\left\{
\begin{array}{lll}
\partial_t u + Au = f(x,u), & t \in J, & x \in \Omega, \\
u = 0, & t \in J, & x \in \partial\Omega, \\
u(t_0, x) = \xi, & & x \in \Omega,
\end{array}
\right.
\tag{12}
$$

satisfying

$$
u(t) = e^{-(t-t_0)A}\xi + \int_{t_0}^{t} e^{-(t-s)A} f(\cdot, u(s)) ds,
$$

where $\xi \in B_{H_0^1}(\hat{u}_0, \varepsilon)$ for $\varepsilon > 0$.

Let $l_k(t)$ $(t \in J)$ be a linear Lagrange basis satisfying $l_k(t_j) = \delta_{kj}$ $(j = 0, 1)$, where δ_{kj} is Kronecker's delta. We define $\omega_0(t)$ as

$$\omega_0(t) = \hat{u}_0 l_0(t) + \hat{u}_1 l_1(t),\ t \in J. \tag{13}$$

In the following, we give a sufficient condition for guaranteeing the existence and the local uniqueness of a mild solution in $B_{L^\infty(J;H_0^1(\Omega))}(\omega_0, \rho)$ for a certain $\rho > 0$.

Theorem 3.2. *We consider the semilinear parabolic equation* (12)*. Let*

$$\delta \geq \left\| \int_{t_0}^{t} e^{-(t-s)A} (\partial_s \omega_0(s) + A\omega_0(s) - f(\cdot, \omega_0(s))) ds \right\|_{L^\infty\left(J;H_0^1(\Omega)\right)}, \tag{14}$$

where ω_0 is defined by (13)*. We assume that there exists a non-decreasing function $L_{\omega_0} : \mathbb{R} \to \mathbb{R}$ such that for $y \in B_{L^\infty(J;H_0^1(\Omega))}(\omega_0, \rho)$*

$$\|f'[y]u\|_{L^\infty(J;L^2(\Omega))} \leq L_{\omega_0}(\rho) \|u\|_{L^\infty(J;H_0^1(\Omega))},\ \forall u \in L^\infty(J; H_0^1(\Omega)), \tag{15}$$

where the function L_{ω_0} depends on ω_0.

If $\rho > 0$ satisfies

$$\varepsilon + \sqrt{\frac{2\pi}{\lambda_{\min} e}}\, \mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) L_{\omega_0}(\rho)\rho + \delta < \rho, \tag{16}$$

then, a mild solution $u(t)$ of (12) *for $t \in J$ uniquely exists in $B_{L^\infty(J;H_0^1(\Omega))}(\omega_0, \rho)$.*

Proof. By using the analytic semigroup e^{-tA}, an operator $\tilde{S} : L^\infty\left(J; H_0^1(\Omega)\right) \to L^\infty\left(J; H_0^1(\Omega)\right)$ is defined by

$$(\tilde{S}z)(t) := e^{-(t-t_0)A}(\xi - \hat{u}_0) + \int_{t_0}^{t} e^{-(t-s)A} g(z(s)) ds, \tag{17}$$

where we put $g(z(t)) := f(\cdot, z(t) + \omega_0(t)) - \partial_t \omega_0(t) - A\omega_0(t)$. We note that $u(t) := z(t) + \omega_0(t)$ is a mild solution of (12) if and only if z is a fixed point of $\tilde{S}$. We derive a condition based on Banach's fixed-point theorem so that $\tilde{S}$ has a fixed-point in $B_{L^\infty(J;H_0^1(\Omega))}(0, \rho)$ for a certain $\rho > 0$.

At first, we derive a condition guaranteeing that $\tilde{S}\left(B_{L^\infty(J;H_0^1(\Omega))}(0, \rho)\right) \subset B_{L^\infty(J;H_0^1(\Omega))}(0, \rho)$ holds. By using (5), Lemma 2.3, and the spectral mapping theorem, the first term in the right-hand side of (17) is estimated by

$$\begin{aligned} \left\| e^{-(t-t_0)A}(\xi - \hat{u}_0) \right\|_{H_0^1} &= \left\| e^{-(t-t_0)A} A^{1/2} (\xi - \hat{u}_0) \right\|_{L^2} \\ &\leq e^{-(t-t_0)\lambda_{\min}} \varepsilon. \end{aligned}$$

Then, we have

$$\left\| e^{-(t-t_0)A}(\xi - \hat{u}_0) \right\|_{L^\infty(J;H_0^1(\Omega))} \leq \varepsilon. \tag{18}$$

Next, we express as $g(z(s)) = g_1(s) + g_2(s)$ with $g_1(s) := f(\cdot, z(s) + \omega_0(s)) - f(\cdot, \omega_0(s))$ and $g_2(s) := f(\cdot, \omega_0(s)) - \partial_t \omega_0(s) - A\omega_0(s)$. From (5) and Lemma 2.4

with $\alpha = \beta = 1/2$, we have

$$\begin{aligned}\left\|\int_{t_0}^{t} e^{-(t-s)A} g_1(s)ds\right\|_{H_0^1} &\leq \int_{t_0}^{t} \|e^{-(t-s)A}\|_{L^2,H_0^1}\|g_1(s)\|_{L^2}ds \\ &= \int_{t_0}^{t} \|A^{\frac{1}{2}}e^{-(t-s)A}\|_{L^2,L^2}\|g_1(s)\|_{L^2}ds \\ &\leq e^{-1/2}\nu(t)\,\|g_1\|_{L^\infty(J;L^2(\Omega))}\,, \end{aligned} \tag{19}$$

where $\nu(t)$ is denoted by

$$\nu(t) := \int_{t_0}^{t} (t-s)^{-1/2} e^{-1/2(t-s)\lambda_{\min}} ds.$$

From (7), the supremum of $\nu(t)$ for $t \in J$ is given by

$$\sup_{t\in J} \nu(t) = \sqrt{\frac{2\pi}{\lambda_{\min}}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right). \tag{20}$$

Fix $s \in J$. For $v \in L^2(\Omega)$ and $0 \leq \theta \leq 1$, it follows from the mean-value theorem that

$$(f(\cdot, \omega_0(s) + z(s)) - f(\cdot, \omega_0(s)), v)_{L^2} = \int_0^1 (f'[\omega_0(s) + \theta z(s)]z(s), v)_{L^2}d\theta.$$

From (15), we obtain

$$\|f(\cdot, z + \omega_0) - f(\cdot, \omega_0)\|_{L^\infty(J;L^2(\Omega))} \leq L_{\omega_0}(\rho)\|z\|_{L^\infty(J;H_0^1(\Omega))}. \tag{21}$$

Therefore, (19) and (20) give

$$\left\|\int_{t_0}^{t} e^{-(t-s)A} g_1(s)ds\right\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \leq \sqrt{\frac{2\pi}{\lambda_{\min}e}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) L_{\omega_0}(\rho)\rho. \tag{22}$$

From (18), (22) and (14) we have

$$\|\tilde{S}(z)\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \leq \varepsilon + \sqrt{\frac{2\pi}{\lambda_{\min}e}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) L_{\omega_0}(\rho)\rho + \delta.$$

The condition (16) yields that $\tilde{S}(B_{L^\infty(J;H_0^1(\Omega))}(0,\rho)) \subset B_{L^\infty(J;H_0^1(\Omega))}(0,\rho)$ holds.

We now show that $\tilde{S}$ becomes a contraction mapping on $B_{L^\infty(J;H_0^1(\Omega))}(0,\rho)$. Let $z_1, z_2 \in B_{L^\infty(J;H_0^1(\Omega))}(0,\rho)$. From the definition of $\tilde{S}$, it follows

$$(\tilde{S}z_1)(t) - (\tilde{S}z_2)(t) = \int_{t_0}^{t} e^{-(t-s)A}\left\{f(\cdot, z_1(s) + \omega_0(s)) - f(\cdot, z_2(s) + \omega_0(s))\right\} ds.$$

Since $z_i + \omega_0 \in B_{L^\infty(J;H_0^1(\Omega))}(\omega_0,\rho)$ $(i = 1, 2)$, we have the following estimate from (5), (15), (20), and Lemma 2.4:

$$\begin{aligned}&\left\|(\tilde{S}z_1) - (\tilde{S}z_2)\right\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \\ &\leq \sqrt{\frac{2\pi}{\lambda_{\min}e}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) L_{\omega_0}(\rho)\|z_1 - z_2\|_{L^\infty\left(J;H_0^1(\Omega)\right)}.\end{aligned}$$

The condition (16) implies

$$\sqrt{\frac{2\pi}{\lambda_{\min}e}}\operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right)L_{\omega_0}(\rho)<1.$$

Then, $\tilde{S}$ becomes a contraction mapping on $B_{L^\infty(J;H_0^1(\Omega))}(0,\rho)$.

Finally, Banach's fixed-point theorem states that there exists a unique fixed-point z in $B_{L^\infty(J;H_0^1(\Omega))}(0,\rho)$. □

Moreover, we obtain the following a posteriori error estimate at $t=t_1$ if Theorem 3.2 holds.

Corollary 3.3. *Under the assumption in Theorem 3.2, let*

$$\tilde{\delta}\geq\left\|\int_{t_0}^{t_1}e^{-(t-s)A}(\partial_s\omega_0(s)+A\omega_0(s)-f(\cdot,\omega_0(s)))ds\right\|_{H_0^1(\Omega)}. \tag{23}$$

Then, the mild solution u of (12) *satisfies*

$$\|u(t_1)-\hat{u}_1\|_{H_0^1}\leq e^{-\tau\lambda_{\min}}\varepsilon+\sqrt{\frac{2\pi}{\lambda_{\min}e}}\operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right)L_{\omega_0}(\rho)\rho+\tilde{\delta}. \tag{24}$$

Proof. Let z be a fixed point of $\tilde{S}$ in the proof of Theorem 3.2. Then,

$$z(t_1)=u(t_1)-\hat{u}_1=e^{-(t_1-t_0)A}(\xi-\hat{u}_0)+\int_{t_0}^{t_1}e^{-(t_1-s)A}g(z(s))ds,$$

where $g(z(s))=f(\cdot,z(s)+\omega_0(s))-A\omega_0(s)-\partial_s\omega_0(s)$. Similar discussions in those in the proof of Theorem 3.2 provide

$$\|u(t_1)-\hat{u}_1\|_{H_0^1}\leq e^{-\tau\lambda_{\min}}\varepsilon+\sqrt{\frac{2\pi}{\lambda_{\min}e}}\operatorname{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right)L_{\omega_0}(\rho)\rho+\tilde{\delta}.$$

□

On the basis of Theorem 3.1, Theorem 3.2, and Corollary 3.3, we provide a verification algorithm for showing the existence of a global-in-time solution in Algorithm 1.

In Algorithm 1, each ball C_{T_k} $(k=1,2,\ldots,n)$ is an enclosure of the solution to (1) for $t\in T_k$. Let us define C_T as

$$C_T:=\left\{y\in L^\infty\left(T;H_0^1(\Omega)\right)\mid y\in C_{T_k},\ k=1,2,\ldots,n\right\}.$$

If Algorithm 1 finishes successfully, we can show that a solution $u(t)$ of (1) for $t\in T$ is enclosed in C_T. Moreover, the solution is asymptotically approaching to ϕ for $t\in(t',\infty)$. Therefore, in this case, the existence of a global-in-time solution to (1) can be proved by verified numerical computations.

Remark. If the global-in-time solution $u(t)$ is enclosed by Algorithm 1, the solution $u(t)\in H_0^1(\Omega)\subset L^2(\Omega)$ for $t\in[0,\infty)$ is expressed by

$$u(t)=e^{-tA}u_0+\int_0^t e^{-(t-s)A}f(\cdot,u(s))ds.$$

The solution u is in $C^0([0,\infty);L^2(\Omega))$. A proof of the assertion is given in Appendix C.

Algorithm 1 Verification algorithm

Set $\hat{\phi} \in V_h$;
Verify the existence and the local uniqueness of a stationary solution ϕ in $B_{H_0^1}(\hat{\phi}, \rho')$;
if Failed in enclosing ϕ **then**
  error (“Failed in enclosing ϕ”);
end if
Set $\hat{u}_0 \in V_h$ and compute ε satisfying $\|u_0 - \hat{u}_0\|_{H_0^1} \le \varepsilon$;
$t' = 0$; $\eta = u_0$; $\hat{u} = \hat{u}_0$; $k = 0$;
while true **do**
  Compute $\|\eta - \phi\|_{H_0^1}$ based on Remark 3.1;
  Choose λ satisfying $0 \le \lambda < \lambda_{\min}/2$;
  if There exists $\rho > 0$ satisfying (11) in Theorem 3.1 **then**
    break;
  end if
  $k = k + 1$;
  $\hat{u}_0 = \hat{u}$; $t_0 = t'$; $\xi = \eta$;
  Set $\tau > 0$. Let $t_1 = t_0 + \tau$ and $T_k = (t_0, t_1]$;
  Choose $\hat{u}_1 \in V_h$ and set $\omega_0(t) = \hat{u}_0 l_0(t) + \hat{u}_1 l_1(t)$ for $t \in T_k$;
  Compute δ defined by (14);
  if there exists $\rho > 0$ satisfying (16) in Theorem 3.2 **then**
    there exists a mild solution $u(t)$ for $t \in (t_0, t_1]$ satisfying (12).
    Define a ball C_{T_k} as $B_{L^\infty(J;H_0^1(\Omega))}(\omega_0, \rho)$ and $\rho_k = \rho$;
    Compute $\tilde{\delta}$ defined by (23);
    Substituting ρ for the right-hand side of (24), update $\varepsilon > 0$ as $\varepsilon = e^{-\tau\lambda_{\min}}\varepsilon + \sqrt{\frac{2\pi}{\lambda_{\min} e}}\,\mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right) L_{\omega_0}(\rho)\rho + \tilde{\delta}$;
  else
    error (“Verification failed for $t \in T_k$.”);
  end if
  $t' = t_1$; $\eta = u(t_1)$; $\hat{u} = \hat{u}_1$;
end while
$n = k$;
disp (“The solution for $t \in (0, \infty)$ exists and $\|u(t) - \phi\|_{H_0^1} \le \rho e^{-\lambda(t-t')}$ holds for $t > t'$ ”);

4. Numerical results

Let $\Omega := \{x = (x_1, x_2) : 0 < x_1, x_2 < 1\} \subset \mathbb{R}^2$ be an unit square domain. We consider the existence of global-in-time solutions for the following semilinear parabolic equations:

$$
\begin{cases}
\partial_t u - \Delta u = f(x, u), & t \in (0, \infty),\ x \in \Omega, \\
u = 0, & t \in (0, \infty),\ x \in \partial\Omega, \\
u(0, x) = 2\sin(\pi x_1)\sin(\pi x_2), & x \in \Omega,
\end{cases}
\tag{25}
$$

where we consider the cases f being given by

$$\begin{aligned}
\text{(Case 1) } f(x,u) &= u^2 + 4\sin(\pi x_1)\sin(\pi x_2),\\
\text{(Case 2) } f(x,u) &= u^2 + 4(\sin(\pi x_1)\sin(\pi x_2) + \sin(2\pi x_1)\sin(2\pi x_2)\\
&\quad + \sin(\pi x_1)\sin(2\pi x_2)),\\
\text{(Case 3) } f(x,u) &= u^2 + 4\sum_{1\le k,l\le 2}\sin(k\pi x_1)\sin(l\pi x_2),
\end{aligned}$$

and

$$\text{(Case 4) } f(x,u) = u^2 + 4\sum_{1\le k,l\le 3}\sin(k\pi x_1)\sin(l\pi x_2).$$

All computations are carried out on CentOS 6.3 with 3.10GHz Intel(R) Xeon(R) CPU E5-2687W, 128GB RAM. We use MATLAB 2012b with INTLAB ver.7.1 [20]. The spectrum method is employed for discretizing the spatial variable. Namely, we construct a numerical solution by using the Fourier basses. For $N \in \mathbb{N}$, a finite dimensional subspace $V_N \subset \mathcal{D}(A)$ is defined by

$$V_N := \left\{ u \in \mathcal{D}(A) \mid u(x,y) = \sum_{k,l=1}^{N} a_{k,l}\sin(k\pi x)\sin(l\pi y),\, a_{k,l} \in \mathbb{R} \right\}.$$

We fix $N = 10$. We set $\tau = 2^{-8}$ and $\lambda = 1/40 (< \lambda_{\min} = 2\pi^2)$ in Algorithm 1. Then, we try to verify the existence of global-in-time solutions to (25) by using Algorithm 1.

Let ϕ denotes a stationary solution of (25). We verify the existence and the local uniqueness of ϕ in a neighborhood of a numerical solution $\hat{\phi} \in V_N$ by using the verification method given in [19]. A radius of the neighborhood is denoted by ρ' satisfying $\|\phi - \hat{\phi}\|_{H_0^1} \le \rho'$. For each case, ρ' is shown in Table 1. The numerical solution $\hat{\phi}$ are displayed in Figure 1, respectively.

TABLE 1. Radii of the neighborhood enclosing ϕ when $N = 10$.

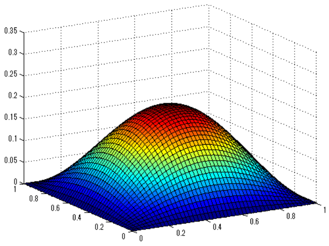
Case	ρ'
1	0.002706328809
2	0.003861742749
3	0.004967902695
4	0.00724564522

For simplicity, in the following we consider (25) for Case 1. Let $\hat{u}_0 \in V_N$ be a numerical approximation of (25) at time $t = t_0$. We give a numerical solution $\hat{u}_1 \in V_N$ of (25) at time $t = t_1$ in Algorithm 1 as follows. We employ the Crank-Nicolson scheme in order to get each $\hat{u}_1 \in V_N$, i.e. we consider the following problem: for $\hat{u}_0 \in V_N$, find $u_1 \in V_N$ such that

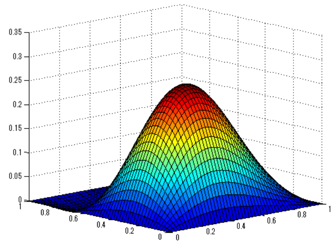
$$\left(\frac{u_1 - \hat{u}_0}{\tau}, v_N\right)_{L^2} + \frac{1}{2}\left(A\hat{u}_0 + Au_1, v_N\right)_{L^2} = \frac{1}{2}\left(f(\cdot,\hat{u}_0) + f(\cdot,u_1), v_N\right)_{L^2},$$

Let $\hat{u}_1 \in V_N$ be a numerical approximation of u_1. We define a numerical solution ω_0 as

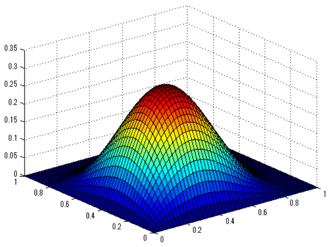
$$\omega_0(t) = \hat{u}_0 l_0(t) + \hat{u}_1 l_1(t),\ t \in T_k \tag{26}$$



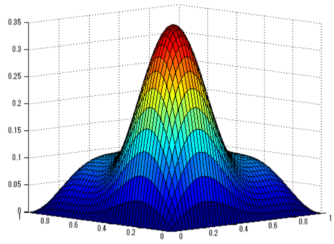
(A) Case 1.



(B) Case 2.



(C) Case 3.



(D) Case 4.

FIGURE 1. The numerical solutions $\hat{\phi}$ of (25) for the four cases.

in Algorithm 1. We compute δ in (14), $\tilde{\delta}$ in (23), $L_\phi(\rho)$ in (10), and $L_{\omega_0}(\rho)$ in (15) for (25) based on estimates in Appendix A and B. Then, Algorithm 1 gives $\rho_k > 0$ satisfying

$$\|u - \omega_0\|_{L^\infty(T_k;H_0^1(\Omega))} \le \rho_k.$$

Figure 2a displays each ρ_k for T_k when $N = 10$ and $\tau = 2^{-8}$.

For Cases $2, 3$, and 4, Figure 2 also shows each ρ_k for T_k when $N = 10$ and $\tau = 2^{-8}$. Furthermore the algorithm 1 gives the following estimates:

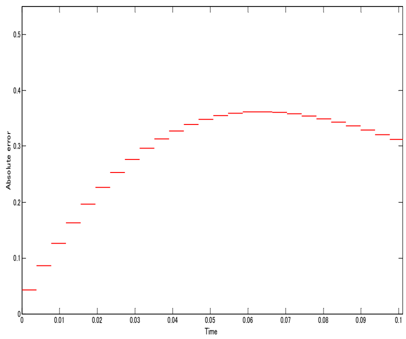
$$\|u(t) - \phi\|_{H_0^1} \le \rho e^{-(t-t')/40}, \ t \in (t', \infty). \tag{27}$$

Table 2 also shows each error estimate ρ and t' of (27).

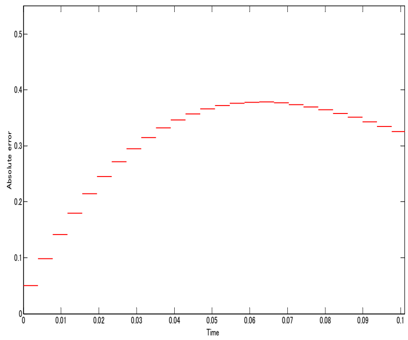
TABLE 2. Error estimates ρ and t' are presented when $N = 10$ and $\tau = 2^{-8}$.

Case	ρ	t'
1	0.973712650429328	0.1015625
2	0.939460907598910	0.10546875
3	0.953394626139478	0.10546875
4	0.954276545574080	0.11328125

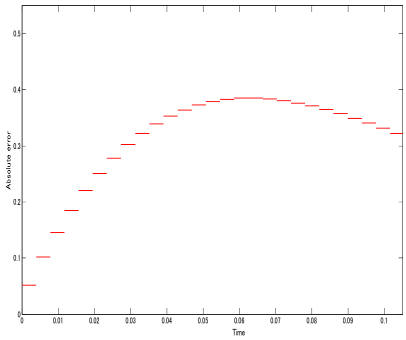
On the other hand, when we consider (25) for Case 1, where we set $u(0, x) = 5.5\sin(\pi x_1)\sin(\pi x_2)$, Algorithm 1 fails in enclosing a global-in-time solution because the existence of the solution $u(t)$ for $t > 0.16796875$ cannot be shown. Figure 3 displays each ρ_k for T_k when $N = 10$ and $\tau = 2^{-8}$. As seen Fig. 3, for this initial



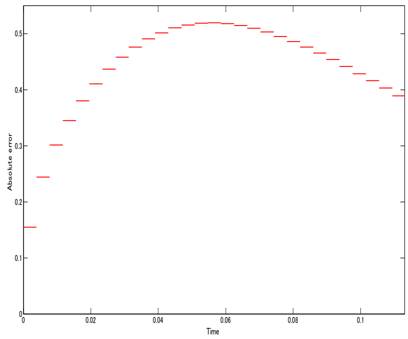
(A) Case 1.



(B) Case 2.



(C) Case 3.



(D) Case 4.

FIGURE 2. Each ρ_k for T_k in (25).

value, as repeatedly using Theorem 3.2 and Corollary 3.3 in Algorithm 1, the error ε in (16) tends to becomes large so that Algorithm 1 cannot verify the existence of a global-in-time solution of (25) for this example.

APPENDIX A. RESIDUAL ESTIMATION

In this Appendix, we show how to estimate δ in (14) and $\tilde{\delta}$ in (23).

For fixed t_0 and t_1 such that $0 \leq t_0 < t_1 < \infty$, let $J = (t_0, t_1]$ and $\tau = t_1 - t_0$. The function space V_h is the same as that in Section 3. For $\hat{u}_0 \in V_h$, we employ the Crank-Nicolson scheme in order to get $\hat{u}_1 \in V_h$, i.e. for $u_0 \in V_h$, we will find $u_1 \in V_h$ such that

$$\left(\frac{u_1 - u_0}{\tau}, v_h\right)_{L^2} + \frac{1}{2}\left(A(u_0 + u_1), v_h\right)_{L^2} = \frac{1}{2}\left(f(\cdot, u_0) + f(\cdot, u_1), v_h\right)_{L^2}$$

for any $v_h \in V_h$. Let $\hat{u}_1 \in V_h$ be a numerical approximation of $u_1 \in V_h$ of this equation replaced u_0 by $\hat{u}_0 \in V_h$. Let l_k $(k = 0, 1)$ be a linear Lagrange basis satisfying $l_k(t_j) = \delta_{k,j}$ $(k, j = 0, 1)$, where $\delta_{k,j}$ is Kronecker's delta. Then, we

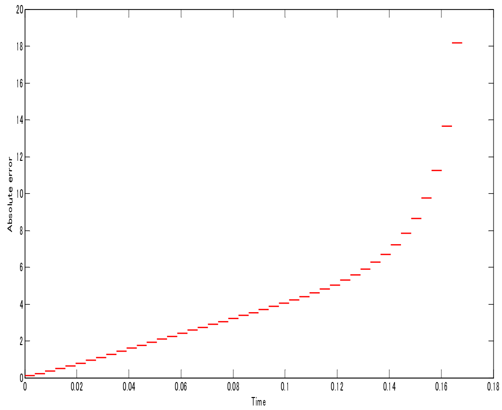


FIGURE 3. Each ρ_k for T_k in (25) for the case 1 and $u(0,x) = 5.5\sin(\pi x_1)\sin(\pi x_2)$.

define $\omega_0 \in L^\infty(J;V_h)$ as

$$\omega_0(t) = \hat{u}_0 l_0(t) + \hat{u}_1 l_1(t),\ t \in J.$$

For a fixed θ satisfying $0 \le \theta \le 1$, we define $\mathcal{C}_\theta \in L^2(\Omega)$ as

$$\mathcal{C}_\theta := \frac{\hat{u}_1 - \hat{u}_0}{\tau} + (1-\theta)A\hat{u}_0 + \theta A\hat{u}_1 - (1-\theta)f(\cdot,\hat{u}_0) - \theta f(\cdot,\hat{u}_1).$$

Let $\Phi(t) := f(\cdot,\hat{u}_1)l_1(t) + f(\cdot,\hat{u}_0)l_0(t)$ for $t \in J$. Then, we have

$$\begin{aligned} &\left\| \int_{t_0}^{t} e^{-(t-s)A}\left(f(\cdot,\omega_0(s)) - \partial_s\omega_0(s) - A\omega_0(s)\right) ds \right\|_{H_0^1} \\ &\le \int_{t_0}^{t} \left\| e^{-(t-s)A}(f(\cdot,\omega_0(s)) - \Phi(s)) \right\|_{H_0^1} ds \\ &+ \int_{t_0}^{t} \left\| e^{-(t-s)A}\left(\Phi(s) - \partial_s\omega_0(s) - A\omega_0(s)\right) \right\|_{H_0^1} ds. \end{aligned} \tag{28}$$

We estimate the first term of (28). Since both $\hat{u}_0$ and $\hat{u}_1$ are in $V_h \subset L^\infty(\Omega)$, a classical error bound of linear interpolation yields for fixed $x \in \Omega$,

$$\begin{aligned} |f(x,\omega_0(t)) - \Phi(t)| &\le \frac{\tau^2}{8}\max_{t\in J}\left|\frac{d^2}{dt^2}f(x,\omega_0(t))\right| \\ &= \frac{\tau^2}{8}\max_{t\in J}\left|f''[\omega_0(t)]\left(\frac{d\omega_0}{dt}\right)^2\right| \\ &= \frac{1}{8}\max_{t\in J}|f''[\omega_0(t)]|\left|(\hat{u}_1 - \hat{u}_0)^2\right|. \end{aligned}$$

From Sobolev's embedding theorem, which will be cited in (33) in Appendix B, it follows

$$\|f(\cdot,\omega_0(t)) - \Phi(t)\|_{L^2} \le \frac{C_{e,4}^2}{8}\|f''[\omega_0]\|_{L^\infty(J;L^\infty(\Omega))}\|\hat{u}_1 - \hat{u}_0\|_{H_0^1}^2. \tag{29}$$

From (29) and Lemma 2.4 with $\alpha = \beta = 1/2$,

$$\begin{aligned}
&\int_{t_0}^{t} \|e^{-(t-s)A}(f(\cdot,\omega_0(s)) - \Phi(s))\|_{H_0^1} ds \\
&= \int_{t_0}^{t} \|A^{1/2}e^{-(t-s)A}(f(\cdot,\omega_0(s)) - \Phi(s))\|_{L^2} ds \\
&\le e^{-1/2}\int_{t_0}^{t} (t-s)^{-1/2} e^{-1/2(t-s)\lambda_{\min}} \|f(\cdot,\omega_0(s)) - \Phi(s)\|_{L^2}\, ds \\
&\le \sqrt{\frac{2\pi}{\lambda_{\min} e}}\, \mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}(t-t_0)}{2}}\right) \|f(\cdot,\omega_0) - \Phi\|_{L^\infty(J;L^2(\Omega))}
\end{aligned}$$

holds. Therefore, we obtain the following upper bound:

$$\begin{aligned}
&\left\| \int_{t_0}^{t} e^{-(t-s)A}(f(\cdot,\omega_0(s)) - \Phi(s)) ds \right\|_{L^\infty\left(J;H_0^1(\Omega)\right)} \\
&\le C_p\alpha^2 \sqrt{\frac{2\pi}{\lambda_{\min} e}}\, \mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right),
\end{aligned}$$

where we put

$$C_p := \frac{C_{e,4}^2}{8}\|f''[\omega_0]\|_{L^\infty(J;L^\infty(\Omega))} \text{ and } \alpha := \|\hat{u}_1 - \hat{u}_0\|_{H_0^1}.$$

We estimate the second term of (28). Since $l_1(s) + l_0(s) = 1$ $(s \in J)$ holds, we have

$$\begin{aligned}
\Phi(s) - \partial_s\omega_0(s) - A\omega_0(s) &= -\left(\mathcal{C}_1 l_1(s) + \mathcal{C}_0 l_0(s)\right) \\
&= -\left((\mathcal{C}_1 - \mathcal{C}_\theta)l_1(s) + (\mathcal{C}_0 - \mathcal{C}_\theta)l_0(s) + \mathcal{C}_\theta\right) \\
&= -\left(\mathcal{C}_\theta + (\mathcal{C}_1 - \mathcal{C}_0)((1-\theta)l_1(s) - \theta l_0(s)\right).
\end{aligned}$$

Then, for a fixed $t \in J$, it sees that

$$\begin{aligned}
&\int_{t_0}^{t} \|e^{-(t-s)A}(\Phi(s) - \partial_s\omega_0(s) - A\omega_0(s))\|_{H_0^1} ds \\
&= \int_{t_0}^{t} \left\|A^{1/2}e^{-(t-s)A}(\Phi(s) - \partial_s\omega_0(s) - A\omega_0(s))\right\|_{L^2} ds \\
&\le \int_{t_0}^{t} e^{-1/2}\|\mathcal{C}_\theta\|_{L^2}(t-s)^{-1/2}e^{-(t-s)\frac{\lambda_{\min}}{2}} ds \\
&+ \|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2} \max_{s\in J} |(1-\theta)l_1(s) - \theta l_0(s))| \int_{t_0}^{t} (t-s)^{-1/2}e^{-(t-s)\frac{\lambda_{\min}}{2}} ds \\
&\le \sqrt{\frac{2\pi}{\lambda_{\min} e}}\, \mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}(t-t_0)}{2}}\right) \left(\|\mathcal{C}_\theta\|_{L^2} + \max(\theta, 1-\theta)\|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2}\right).
\end{aligned}$$

Therefore, when $\theta = 1/2$, both δ and $\tilde{\delta}$ are bounded by

$$\sqrt{\frac{2\pi}{\lambda_{\min} e}}\, \mathrm{erf}\left(\sqrt{\frac{\lambda_{\min}\tau}{2}}\right)\left(C_p\alpha^2 + \left\|\mathcal{C}_{\frac{1}{2}}\right\|_{L^2} + \frac{\|\mathcal{C}_1 - \mathcal{C}_0\|_{L^2}}{2}\right). \tag{30}$$

Here, we sketch a difference between this paper and [14]. In [14] we give a sufficient condition for enclosing a solution to (12) by using an analytic semigroup over $H^{-1}(\Omega)$, where $H^{-1}(\Omega)$ is the topological dual space of $H_0^1(\Omega)$. Let $\langle\cdot,\cdot\rangle$ be a dual product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. A linear operator $\mathcal{A}: H_0^1(\Omega)\to H^{-1}(\Omega)$ is defined by

$$\langle \mathcal{A}u, v\rangle := (\nabla u, \nabla v)_{L^2},\ \forall v\in H_0^1(\Omega).$$

The operator $-\mathcal{A}$ generates an analytic semigroup $\{e^{-t\mathcal{A}}\}_{t\geq 0}$. We define δ_{-1} as

$$\delta_{-1}\geq \left\| \int_{t_0}^{t} e^{-(t-s)\mathcal{A}}(\partial_s\omega_0(s)+\mathcal{A}\omega_0(s)-f(\cdot,\omega_0(s)))ds \right\|_{L^\infty\left(J;H_0^1(\Omega)\right)}.$$

The sufficient condition for enclosing a solution of (12) given in [14] is that there exists $\rho>0$ satisfying

$$\varepsilon + 2\sqrt{\frac{\tau}{e}}L_{\omega_0}(\rho)\rho + \delta_{-1} < \rho, \tag{31}$$

where we recall that ε and L_{ω_0} are given in Theorem 3.2. The main difference of (16) and (31) is δ_{-1}. To estimate δ_{-1}, let us define two functionals $\mathcal{B}(\hat{u}_1)\in H^{-1}(\Omega)$ and $\mathcal{F}(\hat{u}_1)\in H^{-1}(\Omega)$ as

$$\langle \mathcal{B}(\hat{u}_1), v\rangle := \left(\frac{\hat{u}_1-\hat{u}_0}{\tau}, v\right)_{L^2} + (\nabla\hat{u}_1,\nabla v)_{L^2} - (f(\cdot,\hat{u}_1),v)_{L^2}, \forall v\in H_0^1(\Omega),$$
$$\langle \mathcal{F}(\hat{u}_1), v\rangle := \left(\frac{\hat{u}_1-\hat{u}_0}{\tau}, v\right)_{L^2} + (\nabla\hat{u}_1,\nabla v)_{L^2} - (f(\cdot,\hat{u}_0),v)_{L^2}, \forall v\in H_0^1(\Omega),$$

respectively. We obtain

$$\delta_{-1} \leq \frac{1}{4}\sqrt{\frac{|\Omega|\tau}{e}}\|f''[\omega_0]\|_{L^\infty(J;L^\infty(\Omega))}\|\hat{u}_1-\hat{u}_0\|_{L^\infty}^2 + \beta + \left(2+\frac{1-e^{-\tau\lambda_{\min}}}{\tau\lambda_{\min}}\right)\eta, \tag{32}$$

where $|\Omega|$ is the measure of Ω, $\beta=\|\mathcal{B}(\hat{u}_1)\|_{H^{-1}}$, and $\eta=\|\mathcal{B}(\hat{u}_1)-\mathcal{F}(\hat{u}_1)\|_{H^{-1}}$. Here we note that both β and η can be estimated rigorously by using methods given in [17], [18], and [19].

We numerically compare δ with δ_{-1}. We consider (25) for Case 1 with the interval $(0,\infty)$ replaced by $(0,2^{-8}]$. We set a numerical solution ω_0 as (26). Then, we estimate δ and δ_{-1} given in (30) and (32), respectively. The values of δ and δ_{-1} are given in Table 3. Table 3 shows an advantage of the numerical verification

TABLE 3. δ is much smaller than δ_{-1}.

δ	δ_{-1}
0.0402836121	0.82706871027

method based on an analytic semigroup e^{-tA} over $L^2(\Omega)$.

Appendix B. Local Lipschitz Constant

We derive $L_\phi(\rho)$ in (10) and $L_{\omega_0}(\rho)$ in (15) when $f(x,u) = u^2 + g$, where $g \in L^2(\Omega)$ is a given function. Let q be a natural number. There exists Sobolev's embedding constant $C_{e,q} > 0$ satisfying

$$\|u\|_{L^q} \le C_{e,q}\|u\|_{H_0^1},\ \forall u \in H_0^1(\Omega), \tag{33}$$

where $\|\cdot\|_{L^q}$ represents the norm in the usual Lebesgue $L^q(\Omega)$ space [21]. Such a constant $C_{e,p}$ can be numerically estimated (see Lemma 2 in [18] for example). Let J be any interval in $(0,\infty)$. For $\rho > 0$ and a given $v \in L^\infty(J; H_0^1(\Omega))$, let $w \in B_{L^\infty(J;H_0^1(\Omega))}(v,\rho)$. Here, for $u \in L^\infty(J; H_0^1(\Omega))$ and a fixed $s \in J$, we can obtain

$$\begin{aligned}
\|f'[w(s)]u(s)\|_{L^2} &= 2\|w(s)u(s)\|_{L^2} \\
&\le 2\|w(s)\|_{L^4}\|u(s)\|_{L^4} \\
&\le 2C_{e,4}^2\|w(s)\|_{H_0^1}\|u\|_{L^\infty(J;H_0^1(\Omega))} \\
&\le 2C_{e,4}^2(\rho + \|v\|_{L^\infty(J;H_0^1(\Omega))})\|u\|_{L^\infty(J;H_0^1(\Omega))}.
\end{aligned}$$

Therefore, we have

$$L_\phi(\rho) = 2C_{e,4}^2(\rho + \|\phi\|_{L^\infty(J;H_0^1(\Omega))})$$

and

$$L_{\omega_0}(\rho) = 2C_{e,4}^2(\rho + \|\omega_0\|_{L^\infty(J;H_0^1(\Omega))}).$$

Furthermore, we estimate

$$\|\phi\|_{L^\infty(T_k;H_0^1(\Omega))} \le \rho' + \|\hat{\phi}\|_{H_0^1}$$

and

$$\|\omega_0\|_{L^\infty(T_k;H_0^1(\Omega))} \le \max\left\{\|\hat{u}_0\|_{H_0^1}, \|\hat{u}_1\|_{H_0^1}\right\}.$$

Appendix C. The continuity of the global-in-time solution

If the existence of the global-in-time solution to (1) is proved by Algorithm 1, the solution $u(t) \in H_0^1(\Omega) \subset L^2(\Omega)$ for $t \in [0,\infty)$ is expressed by

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(\cdot, u(s))ds.$$

We will show the solution u is in $C^0([0,\infty); L^2(\Omega))$.

First, we will show $f(\cdot,u) \in L^\infty((0,\infty); L^2(\Omega))$. Let $\phi \in H_0^1(\Omega)$ be the stationary solution in Theorem 3.1. Since the global-in-time solution u exponentially converges to ϕ, there exists $\rho > 0$ satisfying $\|u - \phi\|_{L^\infty((0,\infty);H_0^1(\Omega))} \le \rho$. The mean-value theorem and (10) yield

$$\|f(\cdot,u) - f(\cdot,\phi)\|_{L^\infty((0,\infty);L^2(\Omega))} \le L_\phi(\rho)\rho.$$

It follows that

$$\begin{aligned}
\|f(\cdot,u)\|_{L^\infty((0,\infty);L^2(\Omega))} &\le \|f(\cdot,u) - f(\cdot,\phi)\|_{L^\infty((0,\infty);L^2(\Omega))} + \|f(\cdot,\phi)\|_{L^2} \\
&\le L_\phi(\rho)\rho + \|f(\cdot,\phi)\|_{L^2} := M
\end{aligned}$$

Hence, $f(\cdot,u) \in L^\infty((0,\infty); L^2(\Omega))$.

Next, we will show the global-in-time solution u is in $C^0([0,\infty);L^2(\Omega))$. Fix $t' \geq 0$. For $t \in \mathbb{R}$ such that $0 \leq t' < t < \infty$, we have

(34)

$$\begin{aligned}
\|u(t)-u(t')\|_{L^2} \leq{}& \left\|(e^{-tA}-e^{-t'A})u_0\right\|_{L^2} \\
&+ \left\|\int_0^t e^{-(t-s)A}f(\cdot,u(s))ds - \int_0^{t'} e^{-(t'-s)A}f(\cdot,u(s))ds\right\|_{L^2} \\
\leq{}& \|(e^{-tA}-e^{-t'A})u_0\|_{L^2} + \int_{t'}^t \|e^{-(t-s)A}\|_{L^2,L^2}\|f(\cdot,u(s))\|_{L^2}ds \\
&+ \|e^{-(t-t')A}-I\|_{L^2,L^2}\int_0^{t'} \|e^{-(t'-s)A}\|_{L^2,L^2}\|f(\cdot,u(s))\|_{L^2}ds \\
\leq{}& \|(e^{-tA}-e^{-t'A})\|_{L^2,L^2}\|u_0\|_{L^2} + \frac{M(1-e^{-\lambda_{\min}(t-t')})}{\lambda_{\min}} \\
&+ \|e^{-(t-t')A}-I\|_{L^2,L^2}\frac{M(1-e^{-\lambda_{\min}t'})}{\lambda_{\min}},
\end{aligned}$$

where I is an identity operator from $L^2(\Omega)$ to $L^2(\Omega)$ and we have used the spectral mapping theorem. From the continuity of the semigroup, $\|e^{-tA}-e^{-t'A}\|_{L^2,L^2} \to 0$ and $\|e^{-(t-t')A}-I\|_{L^2,L^2} \to 0$ (e.g. [15]) hold if $t \to t'+0$. Then, the right hand side of (34) tends to 0 if $t \to t'+0$. On the other hand, we fix $t' > 0$. For $t \in \mathbb{R}$ such that $0 < t < t' < \infty$, we estimate $\|u(t)-u(t')\|_{L^2}$ by the same way as (34) after exchanging t with t' in (34). By using the continuity of the semigroup, $\|u(t')-u(t)\|_{L^2}$ tends to 0 if $t \to t'-0$. Therefore, the global-in-time solution u is in $C^0([0,\infty);L^2(\Omega))$.

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